9. Metric spaces

- metric spaces
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Metric spaces

Definition 9.1 Let A and B be sets. The **Cartesian product** is the set of tuples defined as

$$A \times B = \{(x, y) \mid x \in A,\ y \in B\}.$$

examples:

- $\{a, b\} \times \{c, d\} = \{(a, c), (a, d), (b, c), (b, d)\}$
- the set $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the Cartesian plane
- the set $[0, 1]^2 = [0, 1] \times [0, 1]$ is a subset of the Cartesian plane bounded by a square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$

Remark 9.2 To denote an element in $\mathbf{R}^n$, we write $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, or simply $x \in \mathbf{R}^n$, where the subscripts $i = 1, \ldots, n$ denote the ith entry of the tuple $(x_1, \ldots, x_n)$ that describes x.

We also simply write $0 \in \mathbf{R}^n$ to mean the point $(0, 0, \ldots 0) \in \mathbf{R}^n$.

Definition 9.3 Let X be a set, and let $d\colon X \times X \to \mathbf{R}$ be a function such that for all $x, y, z \in X$, we have

- $d(x, y) \geq 0$, (*nonnegativity*)
- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$, and (*symmetry*)
- $d(x, z) \leq d(x, y) + d(y, z)$. (*triangle inequality*)

Then the pair (X, d) is called a **metric space**. The function d is called the **metric** or the **distance function**. Sometimes we just write X as the metric space if the metric is clear from context.

Example 9.4 The real numbers $\mathbf{R}$ is a metric space with the metric $d(x, y) = |x - y|$.

proof:

- the first three properties follows immediately from the properties of the absolute value (theorem 2.25)

- to show the triangle inequality, let $x, y, z \in \mathbf{R}$, then we have

$$\begin{aligned} d(x, z) = |x - z| &= |x - y + y - z| \\ &\leq |x - y| + |y - z| = d(x, y) + d(x, z) \end{aligned}$$

Definition 9.5 Let (X, d) be a metric space. A set $S \subseteq X$ is said to be **bounded** if there exists a point $p \in X$ and some number $B \in \mathbf{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

We say (X, d) is bounded if X is a bounded set.

Cauchy-Schwarz inequality

Theorem 9.6 *Cauchy-Schwarz inequality.* Suppose $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, then

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

proof:

$$0 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} (x_i y_j - x_j y_i)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} (x_i^2 y_j^2 - 2x_i y_j x_j y_i + x_j^2 y_i^2)$$

$$= \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{j=1}^{n} y_j^2\right) + \left(\sum_{i=1}^{n} y_i^2\right)\left(\sum_{j=1}^{n} x_j^2\right) - 2\left(\sum_{i=1}^{n} x_i y_i\right)\left(\sum_{j=1}^{n} x_j y_j\right)$$

$$\Longrightarrow \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Theorem 9.7 The function $f\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ given by

$$f(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

is a metric for $\mathbf{R}^n$.

proof: we show that f satisfies the triangle inequality, by theorem 9.6, we have

$$(f(x,z))^2 = \sum_{i=1}^{n} (x_i - z_i)^2 = \sum_{i=1}^{n} (x_i - y_i + y_i - z_i)^2$$

$$= \sum_{i=1}^{n} (x_i - y_i)^2 + 2\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i) + \sum_{i=1}^{n} (y_i - z_i)^2$$

$$\leq \sum_{i=1}^{n}(x_i - y_i)^2 + 2\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 \sum_{i=1}^{n}(y_i - z_i)^2} + \sum_{i=1}^{n}(y_i - z_i)^2$$

$$= \left(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}\right)^2 = (f(x,y) + f(y,z))^2$$

n-dimensional Euclidean space

Definition 9.8 The n-**dimensional Euclidean space** is the metric space $(\mathbf{R}^n, d)$ with the metric d defined by

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \qquad (9.1)$$

Remark 9.9 For $n = 1$, the n-dimensional Euclidean space reduces to the real numbers and the metric given by (9.1) agrees with the standard metric for the set of real numbers $d(x,y) = |x - y|$ in example 9.4.

Open and closed sets

Definition 9.10 Let (X, d) be a metric space, $x \in X$, and $\delta > 0$. Define the **open ball** and **closed ball**, of radius δ around x as

$$B(x,\delta) = \{y \in X \mid d(x,y) < \delta\} \quad \text{and} \quad C(x,\delta) = \{y \in X \mid d(x,y) \leq \delta\},$$

respectively.

Example 9.11 Consider the metric space $\mathbf{R}$, for $x \in \mathbf{R}$ and $\delta > 0$, we have

$$B(x,\delta) = (x-\delta, x+\delta) \quad \text{and} \quad C(x,\delta) = [x-\delta, x+\delta].$$

Example 9.12 Consider the metric space $\mathbf{R}^2$, for $x \in \mathbf{R}^2$ and $\delta > 0$, we have

$$B(x,\delta) = \{y \in \mathbf{R}^2 \mid (x_1 - y_1)^2 + (x_2 - y_2)^2 < \delta^2\}.$$

Definition 9.13 Let (X, d) be a metric space. A subset $V \subseteq X$ is **open** if for all $x \in V$, there exists some $\delta > 0$ such that $B(x, \delta) \subseteq V$. A subset $E \subseteq X$ is **closed** if the complement $E^c = X \setminus E$ is open.

examples:

- $(0, \infty) \subseteq \mathbf{R}$ is open; $[0, \infty) \subseteq \mathbf{R}$ is closed
- $[0, 1) \subseteq \mathbf{R}$ is neither open nor closed
- the singleton $\{x\}$ with $x \in X$ is closed

Theorem 9.14 Let (X, d) be a metric space.

(1) The sets $\emptyset$ and X are open.

(2) If $V_1, \ldots, V_k$ are subsets of X, then $\bigcap_{i=1}^{k} V_i$ is open, $i.e.$, a *finite* intersection of open sets is open.

(3) Let $\{V_i \subseteq X \mid i \in I\}$ be a collection of open subsets of X, where I is an arbitrary index set, then $\bigcup_{i \in I} V_i$ is open, $i.e.$, a union of open sets is open.

proof:

- the sets $\emptyset$ and X are obviously open

- let $x \in \bigcap_{i=1}^{k} V_i$, then $x \in V_1, \ldots, V_k$
  - $V_1, \ldots, V_k$ are open $\implies$ there exists $\delta_1, \ldots, \delta_k > 0$ such that

$$B(x, \delta_1) \subseteq V_1, \ldots, B(x, \delta_k) \subseteq V_k$$

  - choose $\delta = \min\{\delta_1, \ldots, \delta_k\}$, then $B(x, \delta) \subseteq V_1, \ldots, V_k$
    $\implies B(x, \delta) \subseteq \bigcap_{i=1}^{k} V_i$

- let $x \in \bigcup_{i \in I} V_i$, then $\exists V_k \in \{V_i \mid i \in I\}$ such that $x \in V_k$
  - V_k is open $\implies \exists \delta > 0$ such that $B(x, \delta) \subseteq V_k \subseteq \bigcup_{i \in I} V_i$

Theorem 9.15 Let (X, d) be a metric space.

(1) The sets $\emptyset$ and X are closed.

(3) Let $\{V_i \subseteq X \mid i \in I\}$ be a collection of closed subsets of X, where I is an arbitrary index set, then $\bigcap_{i \in I} V_i$ is closed, $i.e.$, an intersection of closed sets is closed.

(2) If $V_1, \ldots, V_k$ are subsets of X, then $\bigcup_{i=1}^{k} V_i$ is closed, $i.e.$, a *finite* union of closed sets is closed.

Remark 9.16 Note that in theorem 9.14, the statement (2) is not true for an arbitrary intersection. For example, the set $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is not open in $\mathbf{R}$.

Similarly, in theorem 9.15, the statement (3) is not true for an arbitrary intersection. For example, consider the set $\bigcup_{n=1}^{\infty}[1/n, \infty) = (0, \infty)$, which is not closed in $\mathbf{R}$.

Theorem 9.17 Let (X, d) be a metric space, $x \in X$, and $\delta > 0$. Then $B(x, \delta)$ is open and $C(x, \delta)$ is closed.

proof: we show that $B(x, \delta)$ is open; let $z \in B(x, \delta)$, then $d(x, z) < \delta$

- choose $\epsilon = \delta - d(x, z)$, let $B(z, \epsilon) = \{y \in X \mid d(y, z) < \epsilon\}$ be an open ball

- let $y \in B(z, \epsilon)$, we have $d(y, z) < \epsilon$, and hence

$$d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \epsilon = d(x, z) + \delta - d(x, z) = \delta$$

$$\implies y \in B(x, \delta) \implies B(z, \epsilon) \subseteq B(x, \delta)$$

Closure and boundary

Definition 9.18 Let (X, d) be a metric space and $A \subseteq X$. The **closure** of A is the set

$$\mathbf{cl}\, A = \bigcap \{E \subseteq X \mid E \text{ is closed and } A \subseteq E\},$$

i.e., $\mathbf{cl}\, A$ is the intersection of all closed sets that contain A.

Definition 9.19 Let (X, d) be a metric space and $A \subseteq X$. The **interior** of A is the set

$$\mathbf{int}\, A = \{x \in A \mid B(x, \delta) \subseteq A \text{ for some } \delta > 0\}.$$

The **boundary** of A is the set

$$\mathbf{bd}\, A = \mathbf{cl}\, A \setminus \mathbf{int}\, A.$$

example: consider $A = (0, 1]$ and $X = \mathbf{R}$, then we have $\mathbf{cl}\, A = [0, 1]$, $\mathbf{int}\, A = (0, 1)$, and $\mathbf{bd}\, A = \{0, 1\}$

Remark 9.20 Notationally, in some textbooks, the closure, interior, and boundary of some set A are denoted as

$$\overline{A} = \mathbf{cl}\, A, \quad A^\circ = \mathbf{int}\, A, \quad \text{and} \quad \partial A = \mathbf{bd}\, A,$$

respectively.

Theorem 9.21 Let (X, d) be a metric space and $A \subseteq X$.

- The closure $\mathbf{cl}\, A$ is closed and $A \subseteq \mathbf{cl}\, A$.

- If A is closed, then $\mathbf{cl}\, A = A$.

proof: let $\mathbf{cl}\, A = \bigcap\{E \subseteq X \mid E \text{ is closed and } A \subseteq E\}$

- the first statement follows directly from the definition of closure and theorem 9.15

- if A is closed, then $A \in \{E \subseteq X \mid E \text{ is closed and } A \subseteq E\} \implies$ $\mathbf{cl}\, A \subseteq A \implies A = \mathbf{cl}\, A$

Theorem 9.22 Let (X, d) be a metric space and $A \subseteq X$, then $x \in \mathbf{cl}\, A$ if and only if for all $\delta > 0$, we have $B(x, \delta) \cap A \neq \emptyset$.

proof: we show the following claim: $x \notin \mathbf{cl}\, A$ if and only if there exists some $\delta > 0$ such that $B(x, \delta) \cap A = \emptyset$

- suppose $x \notin \mathbf{cl}\, A$, then $x \in (\mathbf{cl}\, A)^c$
  - $\mathbf{cl}\, A$ is closed $\implies$ $(\mathbf{cl}\, A)^c$ is open $\implies$ there exists $\delta > 0$ such that $B(x, \delta) \subseteq (\mathbf{cl}\, A)^c \subseteq A^c \implies B(x, \delta) \cap A = \emptyset$
- suppose $\exists \delta > 0$ such that $B(x, \delta) \cap A = \emptyset$, let $x \in X$
  - $B(x, \delta)$ is open $\implies$ $(B(x, \delta))^c$ is closed
  - $B(x, \delta) \cap A = \emptyset \implies A \subseteq (B(x, \delta))^c \implies \mathbf{cl}\, A \subseteq (B(x, \delta))^c$
  - $x \in B(x, \delta) \implies x \notin (B(x, c))^c$
  - put together, we have $x \notin \mathbf{cl}\, A$

Theorem 9.23 Let (X, d) be a metric space and $A \subseteq X$, then $\textbf{int}\, A$ is open and $\textbf{bd}\, A$ is closed.

proof:

- let $x \in \textbf{int}\, A$
  - $x \in \textbf{int}\, A \implies \exists \delta > 0$ such that $B(x, \delta) \subseteq A$
  - let $z \in B(x, \delta)$; $B(x, \delta)$ open $\implies \exists \epsilon > 0$ s.t. $B(z, \epsilon) \subseteq B(x, \delta) \subseteq A$ $\implies z \in \textbf{int}\, A \implies B(x, \delta) \subseteq \textbf{int}\, A \implies \textbf{int}\, A$ is open

- $\textbf{int}\, A$ open $\implies (\textbf{int}\, A)^c$ closed $\implies$

$$\textbf{bd}\, A = \textbf{cl}\, A \setminus \textbf{int}\, A = \textbf{cl}\, A \cap (\textbf{int}\, A)^c$$

  is closed (theorem 9.15)

Theorem 9.24 Let (X, d) be a metric space and $A \subseteq X$, then $x \in \mathbf{bd}\, A$ if and only if for all $\delta > 0$, we have the sets $B(x, \delta) \cap A$ and $B(x, \delta) \cap A^c$ are both nonempty.

proof:

- suppose $x \in \mathbf{bd}\, A$, let $\delta > 0$
  - $x \in \mathbf{bd}\, A \implies x \in \mathbf{cl}\, A$, and hence, by theorem 9.22, we have $B(x, \delta) \cap A \neq \emptyset$
  - assume $B(x, \delta) \cap A^c = \emptyset$, then we have $B(x, \delta) \subseteq A \implies x \in \mathbf{int}\, A$, which is a contradiction

- suppose $B(x, \delta) \cap A \neq \emptyset$ and $B(x, \delta) \cap A^c \neq \emptyset$ for all $\delta > 0$, assume $x \notin \mathbf{bd}\, A$
  - $x \notin \mathbf{bd}\, A \implies x \notin \mathbf{cl}\, A$ or $x \in \mathbf{int}\, A$
  - if $x \notin \mathbf{cl}\, A \implies \exists \delta_0 > 0$ such that $B(x, \delta_0) \cap A = \emptyset$, which is a contradiction
  - if $x \in \mathbf{int}\, A \implies \exists \delta_0 > 0$ such that $B(x, \delta_0) \subseteq A \implies B(x, \delta_0) \cap A^c = \emptyset$, which is a contradiction

Theorem 9.25 Let (X, d) be a metric space and $A \subseteq X$, then $\mathbf{bd}\, A = \mathbf{cl}\, A \cap \mathbf{cl}(A^c)$.

proof: let $x \in \mathbf{bd}\, A$, $\delta > 0$

- by theorem 9.24, we have $B(x, \delta) \cap A$ and $B(x, \delta) \cap A^c$ nonempty
- by theorem 9.22, $B(x, \delta) \cap A \neq \emptyset \implies x \in \mathbf{cl}\, A$ and $B(x, \delta) \cap A^c \neq \emptyset \implies x \in \mathbf{cl}\, A^c$
- hence, we have $\mathbf{bd}\, A = \mathbf{cl}\, A \cap \mathbf{cl}(A^c)$

Sequences in metric spaces

Definition 9.26 A **sequence** in a metric space (X, d) is a function $x\colon \mathbf{N} \to X$. To denote a sequence we write $(x_n)_{n=1}^{\infty}$, where x_n is the nth element in the sequence.

A sequence $(x_n)_{n=1}^{\infty}$ is **bounded** if there exists a point $p \in X$ and $B \in \mathbf{R}$ such that $d(p, x_n) \leq B$ for all $n \in \mathbf{N}$.

Let $(n_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers, then the sequence $(x_{n_i})_{i=1}^{\infty}$ is called a **subsequence** of $(x_n)_{n=1}^{\infty}$.

Definition 9.27 A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to **converge** to a point $p \in X$ if for all $\epsilon > 0$, there exists some $M \in \mathbf{N}$ such that for all $n \geq M$, we have $d(x_n, p) < \epsilon$.

The point p is called a **limit** of $(x_n)_{n=1}^{\infty}$. If the limit p is unique, we write

$$\lim_{n \to \infty} x_n = p.$$

A sequence that converges is said to be **convergent**, and otherwise is **divergent**.

Theorem 9.28 A convergent sequence in a metric space has unique limit.

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proof: let $x, y \in X$ such that $x_n \to x$ and $x_n \to y$; let $\epsilon > 0$

- $x_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \geq M_1$, $d(x_n, x) < \epsilon/2$

- $x_n \to y \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \geq M_2$, $d(x_n, y) < \epsilon/2$

- hence, for all $n \geq M$, we have

$$d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\implies d(x, y) = 0 \implies x = y$$

Theorem 9.29 A convergent sequence in a metric space is bounded.

proof: suppose $x_n \to p \in X$

- let $\epsilon > 0$, $x_n \to p \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $d(x_n, p) < \epsilon$

- choose $B = \max\{d(x_1, p), \ldots, d(x_M, p), \epsilon\}$, then for all $n \in \mathbf{N}$, we have $d(x_n, p) \leq B$

Theorem 9.30 A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) converges to $p \in X$ if and only if there exists a sequence $(a_n)_{n=1}^{\infty}$ of real numbers such that for all $n \in \mathbf{N}$, we have

$$d(x_n, p) \leq a_n \quad \text{and} \quad \lim_{n\to\infty} a_n = 0.$$

proof:

- suppose $x_n \to p$
  - $x_n \to p \implies \forall \epsilon > 0$, $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $d(x_n, p) < \epsilon$ $\implies d(x_n, p) \to 0$
  - choose $a_n = d(x_n, p)$ for all $n \in \mathbf{N}$, then we have $d(x_n, p) \leq a_n$ and $a_n \to 0$
- suppose $a_n \to 0$ with $a_n \in \mathbf{R}$ and $d(x_n, p) \leq a_n$, let $\epsilon > 0$
  - $0 \leq d(x_n, p) \leq a_n$, $a_n \to 0 \implies d(x_n, p) \to 0$ (theorem 3.21)
  - $d(x_n, p) \to 0 \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $d(x_n, p) < \epsilon$ $\implies x_n \to p$

Theorem 9.31 Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (X, d). If $(x_n)_{n=1}^{\infty}$ converges to $p \in X$, then all subsequences of $(x_n)_{n=1}^{\infty}$ converges to p.

proof: let $\epsilon > 0$

- let $x_n \to p$, then $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $d(x_n, p) < \epsilon$

- let $(x_{n_i})_{i=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$, then we have $n_i \geq i$

- hence, for all $i \geq M$, we have $n_i \geq M \implies \forall i \geq M$, $d(x_{n_i}, p) < \epsilon$

Convergence in Euclidean space

Theorem 9.32 Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}^k$, where $x_n \in \mathbf{R}^k$ for all $n \in \mathbf{N}$. Then $(x_n)_{n=1}^{\infty}$ converges if and only if $(x_{n,i})_{n=1}^{\infty}$ converges for all $i = 1, \ldots, k$, *i.e.*,

$$\lim_{n\to\infty} x_n = \left(\lim_{n\to\infty} x_{n,1},\ \ldots,\ \lim_{n\to\infty} x_{n,k}\right).$$

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proof:

- suppose $x_n \to p \in \mathbf{R}^k$, let $\epsilon > 0$; $x_n \to p \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $d(x_n, p) < \epsilon$; hence, $\forall n \geq M$, we have

$$(d(x_n, p))^2 = \sum_{i=1}^{k} (x_{n,i} - p_i)^2 < \epsilon^2 \implies (x_{n,i} - p_i)^2 < \epsilon^2, \quad i = 1, \ldots, k$$

$$\implies |x_{n,i} - p_i| < \epsilon,\ i = 1, \ldots, k \implies x_{n,i} \to p_i \text{ for all } i = 1, \ldots, k$$

- suppose $x_{n,i} \to p_i$ for all $i = 1, \ldots, k$, let $\epsilon > 0$, $p = (p_1, \ldots, p_k)$
  - $x_{n,i} \to p_i$, $i = 1, \ldots, k \implies \exists M_1, \ldots, M_k \in \mathbf{N}$ such that $\forall n \geq M_i$, we have $|x_{n,i} - p_i| < \epsilon/\sqrt{k}$, $i = 1, \ldots, k$
  - choose $M = \max\{M_1, \ldots, M_k\}$, then $\forall n \geq M$, we have

$$d(x_n, p) = \sqrt{\sum_{i=1}^{k} (x_{n,i} - p_i)^2} < \sqrt{\sum_{i=1}^{k} \left(\frac{\epsilon}{\sqrt{k}}\right)^2} = \sqrt{\sum_{i=1}^{k} \frac{\epsilon^2}{k}} = \sqrt{\epsilon^2} = \epsilon$$

$$\implies x_n \to p$$

Convergence properties of topology

Theorem 9.33 Let (X, d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a sequence in X, then $(x_n)_{n=1}^{\infty}$ converges to $p \in X$ if and only if for all open sets $U \subseteq X$ with $p \in U$, there exists some $M \in \mathbf{N}$ such that for all $n \geq M$, we have $x_n \in U$.

proof:

- suppose $x_n \to p$, let $U \subseteq X$ be open and $p \in U$
  - U is an open set contains $p \implies \exists \delta > 0$ such that $B(p, \delta) \subseteq U$
  - $x_n \to p \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $d(x_n, p) < \delta \implies \forall n \geq M$, $x_n \in B(p, \delta) \implies \forall n \geq M$, $x_n \in U$
- suppose for all open sets $U \subseteq X$ with $p \in U$, there exists some $M \in \mathbf{N}$ such that $x_n \in U$ for all $n \geq M$; let $\epsilon > 0$
  - choose $U = B(p, \epsilon)$, then $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $x_n \in B(p, \epsilon)$
  - hence, $\forall n \geq M$, $d(x_n, p) < \epsilon \implies x_n \to p$

Theorem 9.34 Let (X, d) be a metric space, $E \subseteq X$ be a closed set, and $(x_n)_{n=1}^{\infty}$ be a sequence in E that converges to some $p \in X$, then we have $p \in E$.

proof: assume $(x_n)_{n=1}^{\infty}$ in E converges to p but $p \notin E$

- $p \notin E \implies p \in E^c$

- E is closed $\implies$ E^c is open, then by theorem 9.33, $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $x_n \in E^c \implies \forall n \geq M$, $x_n \notin E$, which is a contradiction

Theorem 9.35 Let (X, d) be a metric space and $A \subseteq X$, then $p \in \mathbf{cl}\, A$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ of elements in A such that $\lim_{n\to\infty} x_n = p$.

proof:

- suppose $p \in \mathbf{cl}\, A$, by theorem 9.22, $\forall \delta > 0$, we have $B(p, \delta) \cap A \neq \emptyset$
  - choose $(x_n)_{n=1}^{\infty}$ such that $x_n \in A$ and $d(x_n, p) < \frac{1}{n}$ for all $n \in \mathbf{N}$
  - $0 \leq d(x_n, p) < \frac{1}{n}$ and $\frac{1}{n} \to 0 \implies d(x_n, p) \to 0 \implies x_n \to p$ (theorem 9.30)
- suppose $(x_n)_{n=1}^{\infty}$ in A and $x_n \to p$, let $\delta > 0$
  - $x_n \to p \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $d(x_n, p) < \delta \implies \forall n \geq M$, we have $x_n \in B(p, \delta)$
  - since $x_n \in A$, we have $B(p, \delta) \cap A \neq \emptyset \implies p \in \mathbf{cl}\, A$ (theorem 9.22)

Cauchy sequences and completeness

Definition 9.36 Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is **Cauchy** if for all $\epsilon > 0$, there exists some $M \in \mathbf{N}$ such that for all $n, k \geq M$, we have $d(x_n, x_k) < \epsilon$.

Theorem 9.37 A convergent sequence in a metric space is Cauchy.

proof: let $x_n \to p$, $\epsilon > 0$, then $\exists M \in \mathbf{N}$ such that $\forall n, k \geq M$, we have $d(x_n, p) < \epsilon/2$ and $d(x_k, p) < \epsilon/2$, and hence $\forall n, k \geq M$, we have

$$d(x_n, x_k) \leq d(x_n, p) + d(x_k, p) < \epsilon/2 + \epsilon/2 = \epsilon$$

Definition 9.38 We say a metric space (X, d) is **complete** or **Cauchy-complete** if all Cauchy sequences in X converges to some point in X.

Theorem 9.39 The Euclidean space $\mathbf{R}^k$ is a complete metric space.

proof: let $(x_n)_{n=1}^{\infty}$ be Cauchy with $x_n \in \mathbf{R}^k$ for all $n \in \mathbf{N}$; let $\epsilon > 0$

- $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies$ $\exists M \in \mathbf{N}$ s.t. $\forall m, n \geq M$, $d(x_m - x_n) < \epsilon$
- hence, for all $m, n \geq M$, we have

$$\left(d(x_m, x_n)\right)^2 = \sum_{i=1}^{k} \left(x_{m,i} - x_{n,i}\right)^2 < \epsilon^2$$

$$\implies \left|x_{m,i} - x_{n,i}\right| < \epsilon, \quad i = 1, \ldots, k$$

$\implies$ $\forall i = 1, \ldots, k$, the sequence of real numbers $(x_{n,i})_{n=1}^{\infty}$ is Cauchy

- by theorem 3.45, we have $(x_{n,i})_{n=1}^{\infty}$ converges for all $i = 1, \ldots, k$
- then, by theorem 9.32, we conclude the sequence $(x_n)_{n=1}^{\infty}$ converges