6. Derivative

- definition
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Derivative of functions

Definition 6.1 Let I be an interval, let $f\colon I \to \mathbf{R}$ be a function, and let $c \in I$. We say the function f is **differentiable** at c if the limit

$$L = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. We call L the **derivative** of f at c, and we write $f'(c) = L$.

If f is differentiable at all $c \in I$, then we say the function f is differentiable, and we write f' or $\frac{df}{dx}$ for the function $f'(x)$, $x \in I$.

Example 6.2 Consider the function $f(x) = ax + b$, then $f'(c) = a$ for all $c \in \mathbf{R}$.

proof: let $x, c \in \mathbf{R}$, then we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{ax + b - (ac + b)}{x - c} = \lim_{x \to c} \frac{a(x - c)}{x - c} = \lim_{x \to c} a = a$$

Example 6.3 Consider the function $f(x) = x^2$, then $f'(c) = 2c$ for all $c \in \mathbf{R}$.

proof: let $x, c \in \mathbf{R}$, then we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} \frac{(x + c)(x - c)}{x - c} = \lim_{x \to c}(x + c) = 2c$$

Theorem 6.4 Suppose the function $f \colon I \to \mathbf{R}$ is differentiable at $c \in I$, then f is continuous at c.

proof: f is differentiable at $c \in I \implies$ the limit $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists, and hence,

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c}(x - c) + f(c) \right) = f'(c) \cdot 0 + f(c) = f(c)$$

Remark 6.5 The converse of theorem 6.4 does not hold.

Example 6.6 The function $f(x) = |x|$ is not differentiable at 0.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n = \frac{(-1)^n}{n}$ for all $n \in \mathbf{N}$

- $0 \leq \left|\frac{(-1)^n}{n}\right| \leq \frac{1}{n}$ and $\frac{1}{n} \to 0 \implies x_n \to 0$
- consider the sequence $\left(\frac{f(x_n)-f(0)}{x_n-0}\right)_{n=1}^{\infty}$, we have

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{|x_n|}{x_n} = \frac{\left|\frac{(-1)^n}{n}\right|}{\frac{(-1)^n}{n}} = (-1)^n$$

- $\lim_{n\to\infty} (-1)^n$ does not exist $\implies \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist

Remark 6.7 There exist functions that are continuous but nowhere differentiable.

Differentiation rules

Theorem 6.8 Let I be an interval, let $f\colon I \to \mathbf{R}$ and $g\colon I \to \mathbf{R}$ be differentiable functions at $c \in I$.

- *Linearity.* Let $\alpha \in \mathbf{R}$. Define $h(x) = \alpha f(x) + g(x)$, then $h'(c) = \alpha f'(c) + g'(c)$.

- *Product rule.* Define $h(x) = f(x)g(x)$, then $h'(c) = f'(c)g(c) + f(c)g'(c)$.

- *Quotient rule.* If $g(x) \neq 0$ for all $x \in I$, define $h(x) = f(x)/g(x)$, then

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{\left(g(c)\right)^2}.$$

proof: f, g differentiable at $c \implies \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$, $\lim_{x\to c} \frac{g(x)-g(c)}{x-c}$ exists; f, g continuous at $c \implies \lim_{x\to c} f(x) = f(c)$, $\lim_{x\to c} g(x) = g(c)$

- if $h(x) = \alpha f(x) + g(c)$, then we have

$$\begin{aligned}
\lim_{x\to c} \frac{h(x)-h(c)}{x-c} &= \lim_{x\to c} \frac{\alpha f(x) + g(x) - \alpha f(c) - g(c)}{x-c} \\
&= \alpha \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} = \alpha f'(c) + g'(c)
\end{aligned}$$

- if $h(x) = f(x)g(x)$, then we have

$$\begin{aligned}
\lim_{x\to c} \frac{h(x)-h(c)}{x-c} &= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c) + f(x)g(c) - f(x)g(c)}{x-c} \\
&= \lim_{x\to c} \frac{g(c)(f(x)-f(c)) + f(x)(g(x)-g(c))}{x-c} \\
&= g(c) \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} f(x) \frac{g(x)-g(c)}{x-c} \\
&= f'(c)g(c) + f(c)g'(c)
\end{aligned}$$

- if $h(x) = f(x)/g(x)$, then we have

$$
\begin{aligned}
& \lim_{x\to c} \frac{h(x) - h(c)}{x - c} \\
= \quad & \lim_{x\to c} \frac{f(x)/g(x) - f(c)/g(c)}{x - c} \\
= \quad & \lim_{x\to c} \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x)}{x - c} \\
= \quad & \lim_{x\to c} \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x) + f(x)g(x) - f(x)g(x)}{x - c} \\
= \quad & \lim_{x\to c} \frac{1}{g(x)g(c)} \frac{g(x)(f(x) - f(c)) - f(x)(g(x) - g(c))}{x - c} \\
= \quad & \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}
\end{aligned}
$$

Theorem 6.9 *Chain rule.* Let I_1, I_2 be two intervals. Let $g\colon I_1 \to \mathbf{R}$ be differentiable at $c \in I_1$ and $f\colon I_2 \to \mathbf{R}$ be differentiable at $g(c)$. Define $h\colon I_1 \to \mathbf{R}$ by $h = f \circ g$, then h is differentiable at c, and

$$h'(c) = f'(g(c))g'(c).$$

proof: let $d = g(c)$

- define the following functions:

$$u(y) = \begin{cases} \frac{f(y)-f(d)}{y-d} & y \neq d \\ f'(d) & y = d \end{cases}$$

and

$$v(x) = \begin{cases} \frac{g(x)-g(c)}{x-c} & x \neq c \\ g'(c) & x = c, \end{cases}$$

then we have

$$\lim_{y\to d} u(y) = \lim_{y\to d} \frac{f(y)-f(d)}{y-d} = f'(d) = u(d)$$
$$\lim_{x\to c} v(x) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c) = v(c),$$

$i.e.$, u is continuous at d, v is continuous at c

- note that $f(y)-f(d) = u(y)(y-d)$ and $g(x)-d = v(x)(x-c)$, we have

$$h(x)-h(c) = f(g(x))-f(d) = u(g(x))(g(x)-d) = u(g(x))v(x)(x-c)$$

- put together, we have

$$\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} u(g(x))v(x) = u(g(c))v(c) = f'(g(c))g'(c)$$

Rolle's theorem

Definition 6.10 Let $f\colon S \to \mathbf{R}$ with $S \subseteq \mathbf{R}$.

The function f is said to have a **relative maximum** at $c \in S$ if there exists some $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $f(x) \leq f(c)$.

The function f is said to have a **relative minimum** at $c \in S$ if there exists some $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $f(x) \geq f(c)$.

Theorem 6.11 If the function $f\colon [a,b] \to \mathbf{R}$ has a relative maximum or minimum at $c \in (a,b)$ and f is differentiable at c, then $f'(c) = 0$.

proof: we show the case for c being a relative maximum point

- $c \in (a,b)$ is an relative maximum point $\implies$ $\exists\delta > 0$ such that for all $x \in [a,b]$ and $|x - c| < \delta$, we have $f(x) \leq f(c)$
- let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n = c - \frac{\delta}{2n}$ for all $n \in \mathbf{N}$, then we have $x_n < c$, $x_n \to c$, and $|x_n - c| < \delta$ for all $n \in \mathbf{N}$ $\implies$ $f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0$

- let $(y_n)_{n=1}^{\infty}$ be a sequence with $y_n = c + \frac{\delta}{2n}$ for all $n \in \mathbf{N}$, then we have $y_n > c$, $y_n \to c$, and $|y_n - c| < \delta$ for all $n \in \mathbf{N}$
  $\implies f'(c) = \lim_{n\to\infty} \frac{f(y_n)-f(c)}{y_n-c} \leq 0$

Remark 6.12 In theorem 6.11, the function f does not necessarily have to be defined on a closed interval, but the point c where the relative extremum is achieved has to be on the open interval (a, b).

Remark 6.13 Absolute extremum is a special case of relative extremum.

Theorem 6.14 *Rolle.* Let the function $f\colon [a,b] \to \mathbf{R}$ be continuous and differentiable on (a,b). If $f(a) = f(b)$, then there exists some $c \in (a,b)$ such that $f'(c) = 0$.

proof: let $f(a) = f(b) = K$; f is continuous on $[a,b] \implies$ there exists an absolute maximum point $c_1 \in [a,b]$ and an absolute minimum point $c_2 \in [a,b]$ (theorem 5.33)

- if $c_1 > K$, then $c_1 \in (a,b) \implies f'(c_1) = 0$ (theorem 6.11)
- if $c_2 < K$, then $c_2 \in (a,b) \implies f'(c_2) = 0$ (theorem 6.11)
- if $c_1 = c_2 = K$, then $K \le f(x) \le K$ for all $x \in [a,b] \implies f(x) = K$ for all $x \in [a,b] \implies f'(c) = 0$ for all $c \in (a,b)$

Mean value theorem

Theorem 6.15 *Mean value theorem.* Let the function $f\colon [a,b] \to \mathbf{R}$ be continuous and differentiable on (a,b), then there exists some $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

proof:

- define $g\colon [a,b] \to \mathbf{R}$ with $g(x) = f(x) - f(b) + \frac{f(b)-f(a)}{b-a}(b-x)$
- since $g(a) = g(b) = 0$, by theorem 6.14, there exists $c \in (a,b)$ such that

$$g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$\Longrightarrow \quad f(b) - f(a) = f'(c)(b-a)$$

Theorem 6.16 If the function $f\colon I \to \mathbf{R}$ is differentiable and $f'(x) = 0$ for all $x \in I$, then f is constant.

proof: let $a, b \in I$ with $a < b$, then f is continuous on $[a, b]$ and differentiable on $(a, b) \implies \exists c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a) = 0$$

(since $f'(x) = 0$ for all $x \in I$) $\implies f(b) = f(a)$

Theorem 6.17 Let $f\colon I \to \mathbf{R}$ be a differentiable function.

- The function f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.
- The function f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

proof: we prove the first statement

- suppose $f'(x) \geq 0$ for all $x \in I$, let $a, b \in I$ with $a < b$, then f is continuous on $[a, b]$ and differentiable on $(a, b) \implies \exists c \in (a, b)$ s.t. $f(b) - f(a) = f'(c)(b - a)$ (theorem 6.15) and $f'(c) \geq 0 \implies f(b) - f(a) \geq 0 \implies f(a) \leq f(b)$

- suppose f is increasing, let $c \in I$, then we can find a sequence $(x_n)_{n=1}^{\infty}$ with either $x_n < c$ or $x_n > c$ for all $n \in \mathbf{N}$ such that $x_n \to c$

  - if $x_n < c$ for all $n \in \mathbf{N} \implies f(x_n) \leq f(c)$ for all $n \in \mathbf{N}$, and hence

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0$$

  - if $x_n > c$ for all $n \in \mathbf{N} \implies f(x_n) \geq f(c)$ for all $n \in \mathbf{N}$, and hence

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0$$

  in either case, we have $f'(c) \geq 0$

Taylor's theorem

Definition 6.18 We say the function $f\colon I \to \mathbf{R}$ is n-**times differentiable** on $J \subseteq I$ if $f', f'', \ldots, f^{(n)}$ exist at every point in J, where $f^{(n)}$ denotes the nth derivative of f.

Theorem 6.19 *Taylor.* Suppose the function $f\colon [a,b] \to \mathbf{R}$ is continuous and has n continuous derivatives on $[a,b]$ such that $f^{(n+1)}$ exists on (a,b). Given $x_0, x \in [a,b]$, there exists some $c \in (x_0, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

We denote

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k \quad \text{and} \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

as the n**th order Taylor polynomial** and the n**th order remainder** of f, respectively.

proof: let $x, x_0 \in [a, b]$ and $x \neq x_0$ (if $x = x_0$ then any c satisfies the theorem)

- let $M_{x,x_0} = \frac{f(x)-P_n(x)}{(x-x_0)^{n+1}}$, then we have

$$f(x) = P_n(x) + M_{x,x_0}(x - x_0)^{n+1}$$

- note that for all $0 \leq k \leq n$, we have $f^{(k)}(x_0) = P_n^{(k)}(x_0)$

- let $g(s) = f(s) - P_n(s) - M_{x,x_0}(s - x_0)^{n+1}$, then we have

$$\begin{aligned}
g(x_0) &= f(x_0) - P_n(x_0) - M_{x,x_0}(x_0 - x_0)^{n+1} = 0 \\
g'(x_0) &= f'(x_0) - P_n'(x_0) - M_{x,x_0}(n+1)(x_0 - x_0)^n = 0 \\
&\vdots \\
g^{(n)}(x_0) &= f^{(n)}(x_0) - P_n^{(n)}(x_0) - M_{x,x_0}(n+1)!(x_0 - x_0) = 0
\end{aligned}$$

- by theorem 6.15:

$$\begin{aligned}
g(x_0) = g(x) = 0 \quad &\Longrightarrow \quad \exists x_1 \text{ between } x_0 \text{ and } x \text{ s.t. } g'(x_1) = 0 \\
g'(x_0) = g'(x_1) = 0 \quad &\Longrightarrow \quad \exists x_2 \text{ between } x_0 \text{ and } x_1 \text{ s.t. } g''(x_2) = 0 \\
&\vdots \\
g^{(n-1)}(x_0) = g^{(n-1)}(x_{n-1}) = 0 \quad &\Longrightarrow \quad \exists x_n \text{ between } x_0 \text{ and } x_{n-1} \text{ s.t. } g^{(n)}(x_n) = 0 \\
g^{(n)}(x_0) = g^{(n)}(x_n) = 0 \quad &\Longrightarrow \quad \exists c \text{ between } x_0 \text{ and } x_n \text{ s.t. } g^{(n+1)}(c) = 0
\end{aligned}$$

- note that

$$\frac{d^{n+1}}{ds^{n+1}} M_{x,x_0}(s - x_0)^{n+1} = M_{x,x_0}(n+1)! \quad \text{and} \quad P_n^{(n+1)}(c) = 0$$

- we have the $(n+1)$-times derivative of g at c given by

$$0 = g^{(n+1)}(c) = f^{(n+1)}(c) - M_{x,x_0}(n+1)! \implies M_{x,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}$$

- hence, we have

$$f(x) = P_n(x) + M_{x,x_0}(x - x_0)^{n+1} = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Theorem 6.20 *Second derivative test.* Suppose the function $f\colon (a,b) \to \mathbf{R}$ has two continuous derivatives. If $x_0 \in (a,b)$ such that $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a strict relative minimum at x_0.

proof:

- it is easy to show that f'' is continuous and $f''(x_0) > 0 \implies \exists \delta > 0$ such that $\forall c \in (x_0 - \delta, x_0 + \delta)$, we have $f''(c) > 0$

- then for all $x \in (x_0 - \delta, x_0 + \delta)$, by theorem 6.19, there exists some c_0 between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(c_0)(x - x_0)^2$$

- c_0 between x and $x_0 \implies c_0 \in (x_0 - \delta, x_0 + \delta) \implies f''(c) > 0$, and since $f'(x_0) = 0$, we have

$$f(x) - f(x_0) = \frac{1}{2} f''(c_0)(x - x_0)^2 > 0 \implies f(x) > f(x_0)$$