5. Continuous functions

- cluster points of sets
- limits of functions and sequential properties
- left and right limits
- continuous functions
- operations that preserves continuity
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Cluster points of sets

Definition 5.1 Let $S \subseteq \mathbf{R}$. We say that the point $c \in \mathbf{R}$ is a **cluster point** of S if for all $\delta > 0$, we have $(c-\delta, c+\delta) \cap S \setminus \{c\} \neq \emptyset$, $i.e.$, for all $\delta > 0$, there exists some $x \in S$, such that $0 < |x - c| < \delta$.

examples:

- $S = \{1/n \mid n \in \mathbf{N}\}$ has a cluster point $c = 0$
- $S = (0,1)$ has a set of cluster points given by $[0,1]$
- $S = \mathbf{Q}$ has a set of cluster points given by $\mathbf{R}$
- $S = \{0\}$ has no cluster points
- $S = \mathbf{Z}$ has no cluster points

Theorem 5.2 Let $S \subseteq \mathbf{R}$. Then $c \in \mathbf{R}$ is a cluster point of S if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ of elements in $S \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$.

proof:

- suppose c is a cluster point of S, then $\forall \delta > 0$, $\exists x \in S$ such that $0 < |x - c| < \delta$
  - $\forall n \in \mathbf{N}$, choose $x_n \in S$ such that $0 < |x_n - c| < \frac{1}{n}$
  - $\frac{1}{n} \to 0 \implies |x_n - c| \to 0 \implies x_n \to c$

- suppose there exists a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in S \setminus \{c\}$ for all $n \in \mathbf{N}$ such that $x_n \to c$, let $\delta > 0$
  - $x_n \to c$ with $x_n \in S \setminus \{c\} \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $0 < |x_n - c| < \delta$
  - choose $x = x_M$, then we have $0 < |x - c| < \delta \implies S$ has cluster point c

Limits of functions

Definition 5.3 Let $f \colon S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose there exists an $L \in \mathbf{R}$, and for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. We then say $f(x)$ **converges** to L as x goes to c, and we write

$$f(x) \to L \quad \text{as} \quad x \to c.$$

We say L is a **limit** of $f(x)$ as x goes to c, and if L is unique, we write

$$\lim_{x \to c} f(x) = L.$$

Remark 5.4 The function $f\colon S \to \mathbf{R}$ does not converge to $L \in \mathbf{R}$ as x goes to a cluster point c of S implies that there exists some $\epsilon > 0$, such that for all $\delta > 0$, there exists some $x \in S$ and $0 < |x - c| < \delta$, so that $|f(x) - L| \geq \epsilon$.

Theorem 5.5 Let $f\colon S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. If $f(x) \to L_1$ and $f(x) \to L_2$ as $x \to c$, then $L_1 = L_2$.

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proof: let $\epsilon > 0$

- $f(x) \to L_1$ as $x \to c \implies \exists \delta_1 > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta_1$, $|f(x) - L_1| < \epsilon/2$
- $f(x) \to L_2$ as $x \to c \implies \exists \delta_2 > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta_2$, $|f(x) - L_2| < \epsilon/2$
- choose $\delta = \min\{\delta_1, \delta_2\}$, then for all $x \in S$ and $0 < |x - c| < \delta$, we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |f(x) - L_1| + |f(x) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

$$\implies L_1 = L_2$$

Example 5.6 Let the function $f(x) = ax + b$. Then, for all $c \in \mathbf{R}$, we have $\lim_{x \to c} f(x) = ac + b$.

proof: let $\epsilon > 0$, choose $\delta = \frac{\epsilon}{|a|+1}$, then for all $x \in \mathbf{R}$ and $0 < |x - c| < \delta$, we have

$$|f(x) - (ac + b)| = |(ax + b) - (ac + b)| = |a||x - c| < |a|\delta = \frac{|a|}{|a| + 1}\epsilon \leq \epsilon$$

Example 5.7 Let $f\colon (0, \infty) \to \mathbf{R}$ with $f(x) = \sqrt{x}$. Then, for all $c > 0$, we have $\lim_{x \to c} f(x) = \sqrt{c}$.

proof: let $\epsilon > 0$, choose $\delta = \epsilon\sqrt{c}$, then for all $x > 0$ and $0 < |x - c| < \delta$, we have

$$\begin{aligned} |f(x) - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| &= \left| \frac{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})}{\sqrt{x} + \sqrt{c}} \right| \\ &= \left| \frac{x - c}{\sqrt{x} + \sqrt{c}} \right| \leq \frac{|x - c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} < \epsilon \end{aligned}$$

Example 5.8 Let $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$. Then, $\lim_{x \to 0} f(x) = 1$ ($\neq f(0)$).

proof: let $\epsilon > 0$, choose $\delta = 1$, then $\forall x$ satisfies $0 < |x| < \delta$, we have $x \neq 0 \implies \forall x$ satisfies $0 < |x| < \delta$, we have $|f(x) - 1| = |1 - 1| = 0 < \epsilon$

Theorem 5.9 Let $f \colon S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Then, the following statements are equivalent:

- The function $f(x)$ converges to $L \in \mathbf{R}$ as x goes to c, $i.e.$,

$$\lim_{x \to c} f(x) = L.$$

- For all sequences $(x_n)_{n=1}^{\infty}$ in $S \setminus \{c\}$ such that $\lim_{n \to \infty} x_n = c$, we have $\lim_{n \to \infty} f(x_n) = L$.

proof:

- suppose $\lim_{x\to c} f(x) = L$, let $\epsilon > 0$
  - $\exists\delta > 0$, such that for all $x \in S$ and $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$
  - $x_n \to c$, $x_n \in S \setminus \{c\} \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $0 < |x_n - c| < \delta \implies \forall n \geq M$, we have $|f(x_n) - L| < \epsilon$, *i.e.*, $f(x_n) \to L$

- suppose for all sequences in $S \setminus \{c\}$ s.t. $x_n \to c$, we have $f(x_n) \to L$
  - assume $\lim_{x\to c} f(x) \neq L \implies \exists\epsilon > 0$ s.t. $\forall\delta > 0$, there exists some $x \in S$ and $0 < |x - c| < \delta$, so that $|f(x) - L| \geq \epsilon$
  - choose a sequence $(x_n)_{n=1}^{\infty}$ such that $\forall n \in \mathbf{N}$, $x_n \in S \setminus \{c\}$, $0 < |x_n - c| < \frac{1}{n}$, and $|f(x_n) - L| \geq \epsilon$ for all $n \in \mathbf{N}$
  - however, $\frac{1}{n} \to 0 \implies x_n \to c \implies f(x_n) \to L \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $|f(x_n) - L| < \epsilon$, which is a contradiction

Theorem 5.10 For all $c \in \mathbf{R}$, we have $\lim_{x \to c} x^2 = c^2$.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R} \setminus \{c\}$ such that $x_n \to c$, then according to theorem 3.24, we have $x_n^2 \to c^2 \implies \lim_{x \to c} x^2 = c^2$ (theorem 5.9)

Theorem 5.11 The limit $\lim_{x \to 0} \sin(1/x)$ does not exist, but $\lim_{x \to 0} x \sin(1/x) = 0$.

proof:

- we first show that $\lim_{x \to 0} x \sin(1/x) = 0$: let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R} \setminus \{0\}$ such that $x_n \to 0$; since $\forall n \in \mathbf{N}$, $0 \leq |x_n \sin(1/x_n)| \leq |x_n|$, and $x_n \to 0$, we have $|x_n \sin(1/x_n)| \to 0 \implies \lim_{x \to 0} x \sin(1/x) = 0$

- we now show that $\lim_{x \to 0} \sin(1/x)$ does not exist:
  - choose a sequence $(x_n)_{n=1}^{\infty}$ where $x_n = \frac{2}{(2n-1)\pi}$, then we have $x_n \to 0$

– consider the sequence $(\sin(1/x_n))_{n=1}^{\infty}$, we have

$$\sin(1/x_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n+1}$$

$\implies$ $(\sin(1/x_n))_{n=1}^{\infty}$ does not converge $\implies$ $\lim_{x\to 0}\sin(1/x)$ does not exist

Sequential properties

Theorem 5.12 Let $f, g\colon S \to \mathbf{R}$ be functions and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose $f(x) \leq g(x)$ for all $x \in S$, and we have $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.

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proof: let $(x_n)_{n=1}^{\infty}$ be a sequence in $S \setminus \{c\}$ such that $x_n \to c$

- $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist $\implies$ $(f(x_n))_{n=1}^{\infty}$ and $(g(x_n))_{n=1}^{\infty}$ converges

- let $f(x_n) \to L_1$, $g(x_n) \to L_2$, since $f(x) \leq g(x)$ for all $x \in S$, we have $L_1 \leq L_2$, *i.e.*, $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$

similarly, we can prove the following theorems:

Theorem 5.13 Let $f\colon S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose the limit $\lim_{x\to c} f(x)$ exists, and there exists $a, b \in \mathbf{R}$ such that $a \leq f(x) \leq b$ for all $x \in S \setminus \{c\}$, then $a \leq \lim_{x\to c} f(x) \leq b$.

Theorem 5.14 Let c be a cluster point of $S \subseteq \mathbf{R}$, and $f, g, h \colon S \to \mathbf{R}$ be functions such that $f(x) \leq g(x) \leq h(x)$ for all $x \in S \setminus \{c\}$. Suppose $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$, then

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

Theorem 5.15 Let c be a cluster point of $S \subseteq \mathbf{R}$, and $f, g \colon S \to \mathbf{R}$ be functions such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, we have:

- $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$;
- $\lim_{x\to c}(f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$;
- if $\lim_{x\to c} g(x) \neq 0$ and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Theorem 5.16 Let c be a cluster point of $S \subseteq \mathbf{R}$ and $f\colon S \to \mathbf{R}$ be a function such that $\lim_{x\to c} f(x)$ exists, then we have

$$\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)|.$$

Left and right limits

Definition 5.17 Let $S \subseteq \mathbf{R}$ and $f\colon S \to \mathbf{R}$ be a function.

Suppose c is a cluster point of $S \cap (-\infty, c)$, we say $f(x)$ converges to L as $x \to c^-$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $c - \delta < x < c$, we have $|f(x) - L| < \epsilon$. We call such a limit the **left limit** of f at c, denoted $\lim_{x \to c^-} f(x)$.

Suppose c is a cluster point of $S \cap (c, \infty)$, we say $f(x)$ converges to L as $x \to c^+$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $c < x < c + \delta$, we have $|f(x) - L| < \epsilon$. We call such a limit the **right limit** of f at c, denoted $\lim_{x \to c^+} f(x)$.

Example 5.18 Consider the function f given by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0, \end{cases}$$

then $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 1$, even if $f(0)$ is undefined.

Continuous functions

Definition 5.19 Let $S \subseteq \mathbf{R}$ and $c \in S$. We say the function f is **continuous** at c if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

We say the function f is continuous on the set U for $U \subseteq S$ if f is continuous at every point of U.

Remark 5.20 The function f is not continuous at point $c \in S$ if there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exists some $x \in S$ and $|x - c| < \delta$, so that $|f(x) - f(c)| \geq \epsilon$.

Example 5.21 The function $f(x) = ax + b$ is continuous on $\mathbf{R}$.

proof: let $c \in \mathbf{R}$, $\epsilon > 0$, choose $\delta = \frac{\epsilon}{|a|+1}$, then for all $x \in \mathbf{R}$, $|x - c| < \delta$:

$$|f(x) - f(c)| = |ax + b - ac - b| = |a||x - c| < |a|\delta = \frac{|a|}{|a| + 1}\epsilon \leq \epsilon$$

Example 5.22 The function f given by

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$$

is not continuous at $c = 0$.

proof: choose $\epsilon = 1$ and let $\delta > 0$, then $x = \delta/2$ satisfies $|x| < \delta$, but

$$|f(x) - f(0)| = |1 - 0| = 1 \geq \epsilon$$

Theorem 5.23 Let $S \subseteq \mathbf{R}$ be a set, $c \in S$ be a point, and $f\colon S \to \mathbf{R}$ be a function.

- If c is not a cluster point of S, then the function f is continuous at c.
- If c is a cluster point of S, then the function f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.
- The function f is continuous at c if and only if for all sequences $(x_n)_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = f(c)$.

proof: to show the first statement, let $\epsilon > 0$

- $c \in S$ and c is not a cluster point of $S \implies \exists \delta > 0$ such that

$$(c-\delta, c+\delta) \cap S = \{c\}$$

- then for all $x \in S$ such that $|x - c| < \delta$, we have $x = c$, and hence,

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

we now show the second statement:

- suppose f is continuous at c, let $\epsilon > 0$
  - f is continuous at $c \implies \exists\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$
  - then $\forall x \in S$ such that $0 < |x - c| < \delta$, $|f(x) - f(c)| < \epsilon \implies \lim_{x\to c} f(x) = f(c)$

- suppose $\lim_{x\to c} f(x) = f(c)$, let $\epsilon > 0$
  - $f(x) \to f(c)$ as $x \to c \implies \exists\delta > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$
  - then for all $x \in S$ such that $|x - c| < \delta$: if $x = c$, we have

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

    if $x \neq c$, we have $0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$
  - put together, we conclude that the function f is continuous at c

we now show the third statement

- suppose f is continuous at c, let $(x_n)_{n=1}^{\infty}$ be a sequence in S, $x_n \to c$, let $\epsilon > 0$

  - f is continuous at $c \implies \exists \delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$

  - $x_n \to c \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|x_n - c| < \delta \implies \forall n \geq M$, $|f(x_n) - f(c)| < \epsilon \implies (f(x_n))_{n=1}^{\infty} \to f(c)$

- suppose for all $(x_n)_{n=1}^{\infty}$ in S such that $x_n \to c$, we have $f(x_n) \to f(c)$

  - assume f is not continuous at $c \implies \exists \epsilon > 0$, $\forall \delta > 0$, $\exists x \in S$ such that $|x - c| < \delta$, but $|f(x) - f(c)| \geq \epsilon$

  - choose $x_n \in S$ s.t. $\forall n \in \mathbf{N}$, $0 \leq |x_n - c| < \frac{1}{n}$ but $|f(x_n) - f(x)| \geq \epsilon$

  - $\frac{1}{n} \to 0 \implies x_n \to c \implies f(x_n) \to f(c) \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|f(x_n) - f(c)| < \epsilon$, which is a contradiction

Theorem 5.24 The functions $\sin x$ and $\cos x$ are continuous on $\mathbf{R}$.

proof:

- recall the following properties of $\sin x$ and $\cos x$ for all $x \in \mathbf{R}$:
  - $\sin^2(x) + \cos^2(x) = 1 \implies |\sin x| \leq 1$ and $|\cos x| \leq 1$
  - $|\sin x| \leq |x|$
  - $\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$
  - $\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$

- we first show that $\sin x$ is continuous, let $c \in \mathbf{R}$, let $\epsilon > 0$, choose $\delta = \epsilon$, then for all $x \in \mathbf{R}$ such that $|x - c| < \delta$, we have

$$|\sin x - \sin c| = \left|2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)\right| \leq 2\left|\sin\left(\frac{x-c}{2}\right)\right| \leq 2\frac{|x-c|}{2} = |x-c| < \epsilon$$

- we now show that $\cos x$ is continuous, let $c \in \mathbf{R}$, let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n \to c$, then we have $x_n + \frac{\pi}{2} \to c + \frac{\pi}{2}$, and hence,

$$\lim_{n\to\infty} \cos x_n = \lim_{n\to\infty} \sin\left(x_n + \frac{\pi}{2}\right) = \sin\left(c + \frac{\pi}{2}\right) = \cos c$$

Theorem 5.25 *Dirichlet function.* The Dirichlet function given by

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

is not continuous on all of $\mathbf{R}$.

proof: let $c \in \mathbf{R}$

- if $c \in \mathbf{Q}$, then for all $n \in \mathbf{N}$, there exists $x_n \notin \mathbf{Q}$ s.t. $c < x_n < c + \frac{1}{n}$; $\frac{1}{n} \to 0 \implies x_n \to c$, however,

$$\lim_{n\to\infty} f(x_n) = 0 \neq f(c) = 1$$

$\implies (f(x_n))_{n=1}^{\infty}$ does not converge to $f(c)$

- if $c \notin \mathbf{Q}$, then for all $n \in \mathbf{N}$, there exists $x_n \in \mathbf{Q}$ s.t. $c < x_n < c + \frac{1}{n}$; $\frac{1}{n} \to 0 \implies x_n \to c$, however,

$$\lim_{n\to\infty} f(x_n) = 1 \neq f(c) = 0$$

$\implies (f(x_n))_{n=1}^{\infty}$ does not converge to $f(c)$

Operations that preserves continuity

Theorem 5.26 Let $f, g\colon S \to \mathbf{R}$ be functions on $S \subseteq \mathbf{R}$ and are continuous at $c \in S$.

- The function $f + g$ is continuous at c.
- The function $f \cdot g$ is continuous at c.
- If $g(x) \neq 0$ for all $x \in S$, then the function f/g is continuous at c.

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proof: we show that the function $f + g$ is continuous at c, the other two statements can be proved similarly; let $(x_n)_{n=1}^{\infty}$ be a sequence in S with $x_n \to c$

- f is continuous at $c \implies \lim_{n\to\infty} f(x_n) = f(c)$
- g is continuous at $c \implies \lim_{n\to\infty} g(x_n) = g(c)$
- hence, $\lim_{n\to\infty}(f(x_n) + g(x_n)) = f(c) + g(c) \implies f + g$ is continuous at c

Theorem 5.27 Let $f\colon B \to \mathbf{R}$ and $g\colon A \to B$ be functions on $A, B \subseteq \mathbf{R}$. If g is continuous at $c \in A$ and f is continuous at $g(c) \in B$, then $f \circ g$ is continuous at c.

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proof: let $(x_n)_{n=1}^{\infty}$ be a sequence in A and $x_n \to c \implies g(x_n) \to g(c)$ $\implies f(g(x_n)) \to f(g(c)) \implies f \circ g$ is continuous at c

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Theorem 5.28 Let f be a polynomial function of the form

$$f(x) = a_p x^p + \cdots + a_1 x + a_0.$$

Then, the function f is continuous on $\mathbf{R}$.

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proof: let $c \in \mathbf{R}$, let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}$ and $x_n \to c$, then:

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} (a_p x_n^p + \cdots + a_1 x_n + a_0) \\
&= a_p \lim_{n\to\infty} x_n^p + \cdots + a_1 \lim_{n\to\infty} x_n + a_0 \\
&= a_p c^p + \cdots + a_1 c + a_0 = f(c)
\end{aligned}$$

Example 5.29 Theorems 5.26 and 5.27 allows us to show that some given function is continuous without a huge $\epsilon - \delta$ proof, for example:

- The function $1/x^2$ is continuous on $(0, \infty)$, since x^2 is continuous on $(0, \infty)$.
- The function $(\cos(1/x^2))^2$ is continuous on $(0, \infty)$, since $\cos x$ is continuous on $\mathbf{R}$, and x^2 is continuous on $(0, \infty)$.

Extreme value theorem

Definition 5.30 A function $f\colon S \to \mathbf{R}$ is **bounded** if there exists some $B \geq 0$ such that for all $x \in S$, we have $|f(x)| \leq B$.

Theorem 5.31 If the function $f\colon [a,b] \to \mathbf{R}$ is continuous then f is bounded.

proof:

- suppose f is unbounded, then $\forall B \geq 0$, $\exists x \in [a,b]$ such that $|f(x)| > B$
- let $(x_n)_{n=1}^{\infty}$ be a sequence in $[a,b]$ such that for all $n \in \mathbf{N}$, $|f(x_n)| > n$
- $(x_n)_{n=1}^{\infty}$ is in $[a,b]$ $\implies$ $(x_n)_{n=1}^{\infty}$ is bounded $\implies$ there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ (theorem 3.37) that converges to $c \in \mathbf{R}$

- $a \le x_n \le b \implies a \le x_{n_i} \le b \implies c \in [a, b]$

- f is continuous on $[a, b] \implies f(x_{n_i}) \to f(c) \implies (f(x_{n_i}))_{i=1}^{\infty}$ is bounded

- $|f(x_{n_i})| > n_i \implies (n_i)_{i=1}^{\infty}$ is bounded, which is a contradiction

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Definition 5.32 Let $f\colon S \to \mathbf{R}$ be a function. We say the function f achieves an **absolute minimum** at c if $f(x) \ge f(c)$ for all $x \in S$. We say the function f achieves an **absolute maximum** at d if $f(x) \le f(d)$ for all $x \in S$.

Theorem 5.33 *Extreme value theorem.* Let $f\colon [a,b] \to \mathbf{R}$ be a function on a closed, bounded interval $[a,b]$. If the function f is continuous on $[a,b]$, then f achieves absolute maximum and absolute minimum on $[a,b]$.

proof: we show the case for absolute maximum

- f is continuous on $[a,b] \implies f$ is bounded $\implies$ the set $E = \{f(x) \mid x \in [a,b]\}$ is bounded $\implies \sup E \in \mathbf{R}$ exists

- $\sup E$ is the supremum of $\{f(x) \mid x \in [a,b]\} \implies \forall x \in [a,b]$, $f(x) \le \sup E$, and, there exists some sequence $(f(x_n))_{n=1}^{\infty}$ with $x_n \in [a,b]$ such that $f(x_n) \to \sup E$

- $(x_n)_{n=1}^{\infty}$ is in $[a,b] \implies$ there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ such that $x_{n_i} \to d$ and $d \in [a,b] \implies f(x_{n_i}) \to f(d)$ (since f is continuous)

- $f(x_n) \to \sup E \implies f(x_{n_i}) \to \sup E \implies \sup E = f(d) \implies$ there exists a point $d \in [a,b]$ such that $f(x) \le f(d)$ for all $x \in [a,b]$

Remark 5.34 To apply the extreme value theorem, the function f has to be continuous on a closed, bounded interval.

If the function $f\colon [a,b] \to \mathbf{R}$ is not continuous, consider the function given by

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \text{ or } x = 1 \\ x & x \in (0,1), \end{cases}$$

which neither achieves an absolute maximum nor an absolute minimum on $[0,1]$.

If the function $f\colon S \to \mathbf{R}$ is continuous but S not closed and bounded, consider the function given by

$$f(x) = \frac{1}{x} - \frac{1}{1-x}, \quad S = (0,1),$$

which neither achieves an absolute maximum nor an absolute minimum on $[0,1]$.

Intermediate value theorem

Theorem 5.35 Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$, then there exists some $c \in (a,b)$ such that $f(c) = 0$.

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proof: let $a_1 = a$, $b_1 = b$, for all $n \in \mathbf{N}$, given a_n and b_n, define a_{n+1} and b_{n+1} as:

- $a_{n+1} = a_n$, $b_{n+1} = \frac{a_n+b_n}{2}$, if $f\left(\frac{a_n+b_n}{2}\right) \geq 0$

- $a_{n+1} = \frac{a_n+b_n}{2}$, $b_{n+1} = b_n$, if $f\left(\frac{a_n+b_n}{2}\right) < 0$

then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ has the following properties:

- $a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$ for all $n \in \mathbf{N} \implies (a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are monotone and bounded $\implies (a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge, let $a_n \to c$, $b_n \to d$

- $f(a_n) \leq 0$, $f(b_n) \geq 0$ for all $n \in \mathbf{N}$, since f is continuous, $c, d \in [a,b]$ $\implies \lim_{n\to\infty} f(a_n) = f(c) \leq 0$ and $\lim_{n\to\infty} f(b_n) = f(d) \geq 0$

- $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} = \frac{b_{n-1} - a_{n-1}}{2^2} = \cdots = \frac{b-a}{2^n} \implies b_n - a_n = \frac{b-a}{2^{n-1}}$, and hence, we have

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}\frac{b-a}{2^{n-1}} = 0 = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$

$$\implies \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n \implies c = d$$

put together, we have $f(c) \leq 0$, $f(d) \geq 0$, and $f(c) = f(d)$
$\implies f(c) = f(d) = 0 \implies \exists c \in (a, b)$ such that $f(c) = 0$

Theorem 5.36 *Bolzano's intermediate value theorem.* Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. Suppose $y \in \mathbf{R}$ such that $f(a) < y < f(b)$ or $f(b) < y < f(a)$, then there exists a $c \in (a,b)$ such that $f(c) = y$.

proof: we consider the case for $f(a) < y < f(b)$, the other case is similar

- let $g\colon [a,b] \to \mathbf{R}$ be a function given by $g(x) = f(x) - y$, then g is continuous on $[a,b]$ (theorem 5.26)

- $f(a) < y < f(b) \implies g(a) = f(a) - y < 0$, $g(b) = f(b) - y > 0$
  $\implies \exists c \in (a,b)$ such that $g(c) = f(c) - y = 0$ (theorem 5.35)
  $\implies \exists c \in (a,b)$ such that $f(c) = y$

Theorem 5.37 Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. Suppose the function f achieves absolute minimum at $c \in [a,b]$, and achieves absolute maximum at $d \in [a,b]$. Then, we have $f([a,b]) = [f(c), f(d)]$, *i.e.*, every value between the absolute minimum value and the absolute maximum value is achieved.

proof:

- according to theorem 5.33, we have $f([a,b]) \subseteq [f(c), f(d)]$
- according to theorem 5.36, we have $[f(c), f(d)] \subseteq f([c,d]) \subseteq f([a,b])$
- hence, $f([a,b]) = [f(c), f(d)]$

Remark 5.38 Similarly, theorem 5.36 is false if f is not continuous.

Example 5.39 The polynomial given by $f(x) = x^{2021} + x^{2020} + 9.03x + 1$ has at least one real root.

proof: we have $f(0) = 1 > 0$ and $f(-1) = -8.03 < 0$, hence, by theorem 5.36, there exists some $c \in (-1, 0)$ such that $f(c) = 0$

Uniform continuity

Example 5.40 The function $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$.

proof: let $c \in (0, 1)$ and $\epsilon > 0$, choose $\delta = \min\left\{\frac{c}{2}, \frac{c^2}{2}\epsilon\right\}$, then $\forall x \in (0, 1)$ such that $|x - c| < \delta$, we have

- $||x| - |c|| \le |x - c| < \delta \le \frac{c}{2} \implies -\frac{c}{2} < |x| - c \implies \frac{1}{|x|} < \frac{2}{c}$
- hence, $\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x|c} < \frac{\delta}{|x|c} < \frac{2\delta}{c^2} \le \frac{2}{c^2} \cdot \frac{c^2}{2}\epsilon = \epsilon$

Remark 5.41 Example 5.40 shows that in the definition of function continuity, the number δ can depend on both the number ϵ and the point c.

Definition 5.42 Let $f\colon S \to \mathbf{R}$ be a function. We say the function f is **uniformly continuous** on S if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, c \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

Remark 5.43 In the definition of uniform continuity, the number δ only depends on ϵ.

Example 5.44 The function $f(x) = x^2$ is uniformly continuous on $[0, 1]$.

proof: let $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$, then for all $x, c \in [0, 1]$ and $|x - c| < \delta$, we have $|x + c| \le 2$, and hence,

$$|f(x) - f(c)| = |x^2 - c^2| = |x + c||x - c| < |x + c|\delta \le 2\delta = 2 \cdot \epsilon = \epsilon$$

Remark 5.45 Let $f\colon S \to \mathbf{R}$ be a function. We say the function f is not uniformly continuous on S if there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exists some $x, c \in S$ and $|x - c| < \delta$ so that $|f(x) - f(c)| \geq \epsilon$.

Example 5.46 The function given by $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

proof: choose $\epsilon = 2$, let $\delta > 0$, choose $c = \min\left\{\delta, \frac{1}{2}\right\}$, $x = \frac{c}{2}$, then:

- $x, c \in (0, 1)$ and $|x - c| = \frac{c}{2} \leq \frac{\delta}{2} < \delta$
- $\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x||c|} = \frac{c}{2} \cdot \frac{2}{c^2} = \frac{1}{c} \geq 2 = \epsilon$

Example 5.47 The function $f(x) = x^2$ is not uniformly continuous on $\mathbf{R}$.

proof: choose $\epsilon = 2$, let $\delta > 0$, choose $c = \frac{2}{\delta}$, $x = c + \frac{\delta}{2}$, then we have

- $x, c \in \mathbf{R}$ and $|x - c| = \frac{\delta}{2} < \delta$
- $|x^2 - c^2| = |x + c||x - c| = (2c + \frac{\delta}{2}) \cdot \frac{\delta}{2} = (\frac{4}{\delta} + \frac{\delta}{2}) \cdot \frac{\delta}{2} = 2 + \frac{\delta^2}{4} \geq 2 = \epsilon$

Theorem 5.48 Let $f\colon [a,b] \to \mathbf{R}$ be a function. Then, the function f is continuous on $[a,b]$ if and only if f is uniformly continuous on $[a,b]$.

proof:

- suppose f is uniformly continuous on $[a,b]$: let $c \in [a,b]$, $\epsilon > 0$, then according to uniform continuity, $\exists \delta > 0$ such that for all $x \in [a,b]$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$

- suppose f is continuous on $[a,b]$
  - assume f is not uniformly continuous on $[a,b]$, then $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exists $x, c \in [a,b]$ s.t. $|x - c| < \delta$ but $|f(x) - f(c)| \geq \epsilon$
  - choose sequences $(x_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ such that for all $n \in \mathbf{N}$, $x_n, c_n \in [a,b]$, $|x_n - c_n| < \frac{1}{n}$, but $|f(x_n) - f(c_n)| \geq \epsilon$
  - since $x_n \in [a,b]$ for all $n \in \mathbf{N}$, there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_i} \to c$ and $c \in [a,b]$ (theorem 3.37)

– take subsequence $(c_{n_i})_{i=1}^{\infty}$ of $(c_n)_{n=1}^{\infty}$ according to the indexes n_i of $(x_{n_i})_{i=1}^{\infty}$, then $c_{n_i} \in [a,b]$ for all $n \in \mathbf{N} \implies$ there exists a subsequence $\left(c_{n_{i_j}}\right)_{j=1}^{\infty}$ such that $c_{n_{i_j}} \to d$ and $d \in [a,b]$

– take subsequence $\left(x_{n_{i_j}}\right)_{j=1}^{\infty}$ of $(x_{n_i})_{i=1}^{\infty}$ according to the indexes n_{i_j} of $\left(c_{n_{i_j}}\right)_{j=1}^{\infty}$, then $x_{n_{i_j}} \to c$ since $x_{n_i} \to c$

– $0 \le |x_{n_{i_j}} - c_{n_{i_j}}| < \frac{1}{n_{i_j}}$ and $\frac{1}{n_{i_j}} \to 0 \implies \lim_{j\to\infty} |x_{n_{i_j}} - c_{n_{i_j}}| = 0$ $\implies \lim_{j\to\infty} x_{n_{i_j}} = \lim_{j\to\infty} c_{n_{i_j}} \implies c = d$

– since f is continuous on $[a,b]$ and $x_{n_{i_j}} \to c$, $c_{n_{i_j}} \to c$, we have

$$\lim_{j\to\infty} f(x_{n_{i_j}}) = \lim_{j\to\infty} f(c_{n_{i_j}}) = f(c)$$

$$\implies \quad 0 = |f(c) - f(c)| = \lim_{j\to\infty} |f(x_{n_{i_j}}) - f(c_{n_{i_j}})| \ge \epsilon,$$

which is a contradiction

Lipschitz continuity

Definition 5.49 Let $f\colon S \to \mathbf{R}$ be a function. We say the function f is **Lipschitz continuous** on S if there exists some $K \geq 0$ such that for all $x, y \in S$, we have $|f(x) - f(y)| \leq K|x - y|$.

Remark 5.50 Geometrically, the function f is Lipschitz continuous if and only if all lines intersects the graph of f in at least two distinct points has slope in absolute value less than or equal to K.

Theorem 5.51 Let $f\colon S \to \mathbf{R}$ be a function. If the function f is Lipschitz continuous, then f is uniformly continuous.

proof: let $\epsilon > 0$

- f is Lipschitz continuous $\implies$ $\exists K \geq 0$ such that for all $x, y \in S$, we have $|f(x) - f(y)| \leq K|x - y|$

- choose $\delta = \epsilon/(K+1)$, then for all $x, y \in S$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq K|x - y| < K\delta = \frac{K}{K+1}\epsilon < \epsilon$$

Example 5.52 The function $f(x) = \sqrt{x}$ is Lipschitz continuous on $[1, \infty)$, but is not Lipschitz continuous on $[0, \infty)$.

proof:

- consider the function $f\colon [1, \infty) \to \mathbf{R}$ given by $f(x) = \sqrt{x}$, then $\forall x, y \in [1, \infty)$:
  - $x \geq 1$, $y \geq 1 \implies \sqrt{x} + \sqrt{y} \geq 2$
  - hence,

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}|x - y|$$

$\implies$ f is Lipschitz continuous with $K = 1/2$

- consider the function $g\colon [0, \infty) \to \mathbf{R}$ given by $g(x) = \sqrt{x}$, let $K \geq 0$, choose $x = 0$, $y = \frac{1}{K^2+1}$, then

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \left|\frac{\sqrt{x} - \sqrt{y}}{x - y}\right| = \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} = \sqrt{K^2 + 1} > \sqrt{K^2} = K$$

$$\implies |f(x) - f(y)| > K|x - y|$$