3. Sequences

- sequences and limits
- monotone sequences and subsequences
- inequalities and operations involving limits
- limit superior and limit inferior
- Bolzano-Weierstrass theorem
- Cauchy sequences

Sequences and limits

Definition 3.1 A **sequence** (of real numbers) is a function $x\colon \mathbf{N} \to \mathbf{R}$. To denote a sequence we write $(x_n)_{n=1}^{\infty}$, where x_n is the nth element in the sequence.

- sequence need not start at $n = 1$, $e.g.$, the sequence $x\colon \{n \in \mathbf{Z} \mid n \geq m\} \to \mathbf{R}$ is denoted $(x_n)_{n=m}^{\infty}$

Definition 3.2 A sequence $(x_n)_{n=1}^{\infty}$ is **bounded** if there exists some $B \geq 0$ such that $|x_n| \leq B$ for all $n \in \mathbf{N}$.

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examples:

- the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is bounded since $\frac{1}{n} \leq 1$ for all n

- the sequence $(n)_{n=1}^{\infty}$ is not bounded since for all $B \geq 0$ there exists some $n \geq B$ according to the Archimedean property

Definition 3.3 A sequence $(x_n)_{n=1}^{\infty}$ is said to **converge** to $x \in \mathbf{R}$ if for all $\epsilon > 0$, there exists an $M \in \mathbf{N}$ such that for all $n \geq M$, we have $|x_n - x| < \epsilon$. The number x is called a **limit** of the sequence. If the limit x is unique, we write

$$x = \lim_{n \to \infty} x_n.$$

A sequence that converges is said to be **convergent**, and otherwise is **divergent**.

Remark 3.4 A sequence $(x_n)_{n=1}^{\infty}$ is divergent if for all $x \in \mathbf{R}$, there exists some $\epsilon > 0$, such that for all $M \in \mathbf{N}$, there exists an $n \geq M$, so that $|x_n - x| \geq \epsilon$.

Theorem 3.5 Let $x, y \in \mathbf{R}$. If for all $\epsilon > 0$, $|x - y| < \epsilon$, then $x = y$.

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proof: assume $x \neq y \implies |x-y| > 0$; take $\epsilon = \frac{1}{2}|x-y|$
$\implies |x-y| < \frac{1}{2}|x-y| \implies |x-y| < 0$, which is a contradiction

Theorem 3.6 If $(x_n)_{n=1}^{\infty}$ converges to x and y, then $x = y$, *i.e.*, a convergent sequence has a unique limit.

proof: let $\epsilon > 0$

- $(x_n)_{n=1}^{\infty}$ converges to $x \implies \exists M_1 \in \mathbf{N}$, $\forall n \geq M_1$, $|x_n - x| < \epsilon/2$
- $(x_n)_{n=1}^{\infty}$ converges to $y \implies \exists M_2 \in \mathbf{N}$, $\forall n \geq M_2$, $|x_n - y| < \epsilon/2$
- let $M = M_1 + M_2$, then $M \geq M_1$ and $M \geq M_2$, then we have

$$|x_M - x| < \epsilon/2 \quad \text{and} \quad |x_M - y| < \epsilon/2,$$

hence,

$$\begin{aligned} |x - y| &= |(x - x_M) + (x_M - y)| \\ &\leq |x - x_M| + |y - x_M| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

- according to theorem 3.5, we have $x = y$

Remark 3.7 Sometimes we write '$x_n \to x$ as $n \to \infty$' to mean $x = \lim_{n\to\infty} x_n$. We may also avoid the 'as $n \to \infty$' part if the limiting process is clear from the context.

Example 3.8 Given the sequence $(x_n)_{n=1}^{\infty}$ with $x_n = c \in \mathbf{R}$ for all $n \in \mathbf{N}$, we have $\lim_{n\to\infty} x_n = c$.

proof: let $\epsilon > 0$, $M = 1$, then for all $n \geq M$, we have

$$|x_n - c| = |c - c| = 0 < \epsilon$$

Example 3.9 The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to $x = 0$, $i.e.$, $\lim_{n\to\infty} \frac{1}{n} = 0$.

proof: let $\epsilon > 0$, choose an $M \in \mathbf{N}$ such that $M > 1/\epsilon$ (such an M exists according to the Archimedian property), then for all $n \geq M$, we have
$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| \leq \frac{1}{M} < \epsilon$

Example 3.10 The sequence $\left(\frac{1}{n^2+2n+100}\right)_{n=1}^{\infty}$ converges to $x = 0$.

proof: let $\epsilon > 0$ choose $M \in \mathbf{N}$ such that $M \geq \epsilon^{-1}/2$, then for all $n \geq M$, we have

$$\left|\frac{1}{n^2 + 2n + 100} - 0\right| = \frac{1}{n^2 + 2n + 100} \leq \frac{1}{2n} \leq \frac{1}{2M} < \epsilon$$

Example 3.11 The sequence $(x_n)_{n=1}^{\infty}$ where $x_n = (-1)^n$ is divergent.

proof: let $x \in \mathbf{R}$, $M \in \mathbf{N}$, then

$$\begin{aligned} & |x_M - x_{M+1}| = \left|(-1)^M - (-1)^{M+1}\right| = 2 \\ \Longrightarrow \quad & 2 = |(x_M - x) + (x - x_{M+1})| \leq |x_M - x| + |x_{M+1} - x| \\ \Longrightarrow \quad & |x_M - x| \geq 1 \quad \text{or} \quad |x_{M+1} - x| \geq 1, \end{aligned}$$

$i.e.$, let $\epsilon = 1$, $n = M$, we have either $|x_n - x| \geq \epsilon$ or $|x_{n+1} - x| \geq \epsilon$

Theorem 3.12 If $(x_n)_{n=1}^{\infty}$ is convergent, then $(x_n)_{n=1}^{\infty}$ is bounded.

proof:

- suppose $(x_n)_{n=1}^{\infty}$ converges to x, let $\epsilon = 1$, then there exists some $M \in \mathbf{N}$ such that for all $n \geq M$, $|x_n - x| < 1 \implies x_n < |x| + 1$

- let $B = \max\{|x_1|, |x_2|, \ldots, |x_M|, |x| + 1\}$, since $x_n \leq |x_n|$ for all $n \in \mathbf{N}$, $n \leq M$, and $x_n < |x| + 1$ for all $n \geq M$, we have $B \geq |x_n|$ for all $n \in \mathbf{N}$

Monotone sequences

Definition 3.13

- A sequence $(x_n)_{n=1}^{\infty}$ is **monotone increasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbf{N}$.
- A sequence $(x_n)_{n=1}^{\infty}$ is **monotone decreasing** if $x_n \geq x_{n+1}$ for all $n \in \mathbf{N}$.
- If $(x_n)_{n=1}^{\infty}$ is either monotone increasing or monotone decreasing, we say the sequence $(x_n)_{n=1}^{\infty}$ is **monotone** (or monotonic).

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examples:

- the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is monotone decreasing
- the sequence $\left(-\frac{1}{n}\right)_{n=1}^{\infty}$ is monotone increasing
- the sequence $((-1)^n)_{n=1}^{\infty}$ is not monotone

Theorem 3.14 A monotone sequence $(x_n)_{n=1}^{\infty}$ converges if and only if it is bounded.

- If the sequence $(x_n)_{n=1}^{\infty}$ is monotone increasing and bounded, then

$$\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbf{N}\}.$$

- If the sequence $(x_n)_{n=1}^{\infty}$ is monotone decreasing and bounded, then

$$\lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbf{N}\}.$$

proof: we prove for monotone increasing sequences

- suppose $(x_n)_{n=1}^{\infty}$ is convergent, according to theorem 3.12, it is bounded
- suppose $(x_n)_{n=1}^{\infty}$ is monotone increasing and bounded
  - $(x_n)_{n=1}^{\infty}$ is monotone increasing $\implies$ $x_n \le x_{n+1}$ for all $n \in \mathbf{N}$

- $(x_n)_{n=1}^{\infty}$ is bounded $\implies$ the set $\{x_n \mid n \in \mathbf{N}\}$ has supremum $x = \sup\{x_n \mid n \in \mathbf{N}\}$

- let $\epsilon > 0$, according to theorem 2.17, there exists some $M \in \mathbf{N}$ such that $x - \epsilon < x_M \leq x$, then for all $n \geq M$, we have

$$x - \epsilon < x_M \leq x_n \leq x < x + \epsilon \implies |x_n - x| < \epsilon$$

Example

recall the following lemma from example 1.8:

Lemma 3.15 *Bernoulli's inequality.* If $x \geq -1$ then $(x+1)^n \geq 1 + nx$ for all $n \in \mathbf{N}$.

Theorem 3.16 If $c \in (0,1)$ then the sequence $(c^n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} c^n = 0$. If $c > 1$, the sequence $(c^n)_{n=1}^{\infty}$ does not converge.

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proof:

- if $c > 1$, we show that the sequence $(c^n)_{n=1}^{\infty}$ is unbounded (and hence does not converge):
  - let $B \geq 0$, then there exists some $n \in \mathbf{N}$, $n > \frac{B}{c-1}$ such that

$$c^n = ((c-1)+1)^n \geq 1 + n(c-1) > n(c-1) > B$$

(the first inequality is because of lemma 3.15)

- if $c \in (0,1)$, we first show that $(c^n)_{n=1}^{\infty}$ is monotone decreasing and bounded (and hence, convergent), $i.e.$, show that $c^{n+1} \leq c^n \leq c$ for all $n \in \mathbf{N}$ by induction:
  - suppose $n = 1 \implies c^2 \leq c \leq c$, the first inequality holds since $0 < c < 1$
  - suppose $n > 1$, and $c^{n+1} \leq c^n \leq c$, then we have

$$c^{n+2} \leq c^{n+1} \leq c^n \leq c$$

  let $\lim_{n\to\infty} c^n = L$, we now show that $L = 0$:
  - let $\epsilon > 0$, then there exists some $M \in \mathbf{N}$ such that for all $n \geq M$ such that

$$|c^n - L| < \frac{1}{2}(1-c)\epsilon$$

- hence, we have

$$\begin{aligned}
(1-c)|L| &= |L - cL| \\
&= |(L - c^{M+1}) + (c^{M+1} - cL)| \\
&\leq |L - c^{M+1}| + c|c^M - L| \\
&< |L - c^{M+1}| + |c^M - L| \\
&< \frac{1}{2}(1-c)\epsilon + \frac{1}{2}(1-c)\epsilon \\
&= (1-c)\epsilon,
\end{aligned}$$

$i.e.$, $|L| < \epsilon$ for all $\epsilon > 0$ (according to theorem 2.14) $\implies |L| \leq 0$ $\implies L = 0$

Subsequences

Definition 3.17 Let $(x_n)_{n=1}^{\infty}$ be a sequence and $(n_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers. The sequence $(x_{n_i})_{i=1}^{\infty}$ is called a **subsequence** of $(x_n)_{n=1}^{\infty}$.

example: consider the sequence $(x_n)_{n=1}^{\infty} = (n)_{n=1}^{\infty}$, *i.e.*, $1, 2, 3, 4, \ldots$

- the following are subsequences of $(x_n)_{n=1}^{\infty}$:
  - $1, 3, 5, 7, 9, 11, \ldots$, described with $(x_{n_i})_{i=1}^{\infty} = (x_{2i-1})_{i=1}^{\infty}$
  - $2, 4, 6, 8, 10, 12, \ldots$, described with $(x_{n_i})_{i=1}^{\infty} = (x_{2i})_{i=1}^{\infty}$
  - $2, 3, 5, 7, 11, 13, \ldots$, described with $(x_{n_i})_{i=1}^{\infty}$ where n_i are primes

- the following are not subsequences of $(x_n)_{n=1}^{\infty}$:
  - $1, 1, 1, 1, 1, 1, \ldots$
  - $1, 1, 3, 3, 5, 5, \ldots$

Theorem 3.18 If $\lim_{n\to\infty} x_n = x$, then all subsequences of $(x_n)_{n=1}^{\infty}$ converge to x.

proof:

- let $(x_{n_i})_{i=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$
- let $\epsilon > 0$, then there exists some $M_0 \in \mathbf{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq M_0$
- let $M = M_0$, then for all $i \geq M$, since $n_i \geq i \geq M = M_0$, we have

$$|x_{n_i} - x| < \epsilon$$

Remark 3.19 Theorem 3.18 implies that the sequence $((-1)^n)_{n=1}^{\infty}$ is divergent.

Inequalities involving limits

Theorem 3.20 The sequence $(x_n)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} x_n = x$ if and only if $(|x_n - x|)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} |x_n - x| = 0$.

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proof: let $\epsilon > 0$

- suppose $\lim_{n\to\infty} x_n = x$, then $\exists M_0 \in \mathbf{N}$ such that $\forall n \geq M_0$, $|x_n - x| < \epsilon$; let $M = M_0$, then $\forall n \geq M = M_0$, we have

$$|x_n - x - 0| = |x_n - x| < \epsilon$$

- suppose $\lim_{n\to\infty} |x_n - x| = 0$, then $\exists M \in \mathbf{N}$, $\forall n \geq M$, we have $|x_n - x - 0| < \epsilon$, *i.e.*, $|x_n - x| < \epsilon$

Theorem 3.21 *Squeeze theorem.* Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, and $(x_n)_{n=1}^{\infty}$ be sequences such that

$$a_n \leq x_n \leq b_n$$

for all $n \in \mathbf{N}$. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge and

$$\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n.$$

Then $(x_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} x_n = x$.

proof: let $\epsilon > 0$

- $a_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \geq M_1$, $|a_n - x| < \epsilon$
- $b_n \to x \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \geq M_2$, $|b_n - x| < \epsilon$
- $a_n \leq x_n \leq b_n \implies a_n - x \leq x_n - x \leq b_n - x$
- take $M = \max\{M_1, M_2\}$, then $\forall n \geq M$, we have

$$-\epsilon < a_n - x \leq x_n - x \leq b_n - x < \epsilon \implies |x_n - x| < \epsilon$$

Example 3.22 The sequence $\left(\frac{n^2}{n^2+n+1}\right)_{n=1}^{\infty}$ converges with

$$\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1.$$

proof:

- let $\epsilon > 0$, we have

$$0 \le \left|\frac{n^2}{n^2+n+1} - 1\right| = \left|\frac{n+1}{n^2+n+1}\right| \le \frac{n+1}{n^2+n} = \frac{1}{n}$$

- $0 \to 0$ and $\frac{1}{n} \to 0 \implies \left|\frac{n^2}{n^2+n+1} - 1\right| \to 0 \implies \frac{n^2}{n^2+n+1} \to 1$

Theorem 3.23 Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be sequences.

- If $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ converge and $x_n \le y_n$ for all $n \in \mathbf{N}$, then we have
$$\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n.$$
- If $(x_n)_{n=1}^\infty$ converges and $a \le x_n \le b$ for all $n \in \mathbf{N}$, then
$$a \le \lim_{n\to\infty} x_n \le b.$$

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proof: we show the first statement; the second statement can then be proved by considering $(y_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ where $y_n = a \le x_n \le b = z_n$

- let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, suppose $x > y$
- $x > y \implies x - y > 0$, let $\epsilon = \frac{x-y}{2} > 0$
- $x_n \to x \implies \exists M_1 \in \mathbf{N}$ s.t. $\forall n \ge M_1$, $|x_n - x| < \frac{x-y}{2}$
- $y_n \to y \implies \exists M_2 \in \mathbf{N}$ s.t. $\forall n \ge M_2$, $|y_n - y| < \frac{x-y}{2}$

- let $M = \max\{M_1, M_2\}$, we have $x_M - x > -\frac{x-y}{2}$ and $y_M - y < \frac{x-y}{2}$, hence,
$$x_M > x - \frac{x-y}{2} = \frac{x+y}{2} = y + \frac{x-y}{2} > y_M,$$
which contradicts to $x_n \leq y_n$ for all $n \in \mathbf{N}$

Operations involving limits

Theorem 3.24 Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

- The sequence $(x_n + y_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(x_n+y_n) = x+y$.

- For all $c \in \mathbf{R}$, the sequence $(cx_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} cx_n = cx$.

- The sequence $(x_n y_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n y_n = xy$.

- If $y_n \neq 0$ for all $n \in \mathbf{N}$ and $y \neq 0$, then the sequence $\left(\frac{x_n}{y_n}\right)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

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proof:

- to show $x_n \to x$, $y_n \to y \implies x_n + y_n \to x + y$, let $\epsilon > 0$
  - $x_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \geq M_1$, $|x_n - x| < \epsilon/2$
  - $y_n \to y \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \geq M_2$, $|y_n - y| < \epsilon/2$

– let $M = \max\{M_1, M_2\}$, then for all $n \geq M$, we have

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \epsilon/2 + \epsilon/2 = \epsilon$$

- to show $x_n \to x \implies cx_n \to cx$, let $\epsilon > 0$
  – $x_n \to x \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|x_n - x| < \frac{1}{|c|+1}\epsilon$
  – then for all $n \geq M$, we have $|cx_n - cx| = |c||x_n - x| < \frac{|c|}{|c|+1}\epsilon < \epsilon$

- we show that $x_n \to x$, $y_n \to y \implies x_n y_n \to xy$:
  – $x_n \to x \implies |x_n - x| \to 0$
  – $y_n \to y \implies |y_n - y| \to 0$, and $(y_n)_{n=1}^{\infty}$ is bounded, *i.e.*, $\exists B \geq 0$, $|y_n| \leq B$

- hence, we have

$$\begin{aligned} 0 \le |x_n y_n - xy| &= |x_n y_n + x y_n - x y_n - xy| \\ &= |(x_n - x) y_n + (y_n - y) x| \\ &\le |x_n - x||y_n| + |y_n - y||x| \\ &\le |x_n - x| B + |y_n - y||x| \end{aligned}$$

- according to the previous statements, we have

$$|x_n - x| \to 0 \implies |x_n - x| B \to 0$$

  and

$$|y_n - y| \to 0 \implies |y_n - y||x| \to 0,$$

  then $|x_n - x|B + |y_n - y||x| \to 0$

- hence, according to theorem 3.21, $|x_n y_n - xy| \to 0$

- to prove $x_n \to x$, $y_n \to y$ ($y_n \neq 0$ for all $n \in \mathbf{N}$, $y \neq 0$) $\implies \frac{x_n}{y_n} \to \frac{x}{y}$, we first show that there exists some $b > 0$ such that $|y_n| \geq b$:
  - let $\epsilon = \frac{|y|}{2}$, then $y_n \to y \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $|y_n - y| < \frac{|y|}{2}$
  - then for all $n \geq M$, we have

$$\frac{|y|}{2} > |y_n - y| \geq ||y_n| - |y|| \implies |y_n| > \frac{|y|}{2}$$

    (the second inequality is from the reverse triangle inequality)
  - take $b = \min\{|y_1|, \ldots, |y_M|, |y|/2\}$, we have $|y_n| \geq b$ for all $n \in \mathbf{N}$

  we then show that $\left(\frac{1}{y_n}\right)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$: note that

$$0 \leq \left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y_n - y}{y_n y}\right| = \frac{|y_n - y|}{|y_n||y|} \leq \frac{|y_n - y|}{b|y|},$$

  and $y_n \to y \implies \frac{|y_n - y|}{b|y|} \to 0$, hence, $\left|\frac{1}{y_n} - \frac{1}{y}\right| \to 0$, *i.e.*, $\frac{1}{y_n} \to \frac{1}{y}$

  put together, $x_n \to x$ and $\frac{1}{y_n} \to \frac{1}{y} \implies \frac{x_n}{y_n} \to \frac{x}{y}$

Theorem 3.25 If $(x_n)_{n=1}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} x_n = x$, and $x_n \geq 0$ for all $n \in \mathbf{N}$, then the sequence $\left(\sqrt{x_n}\right)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

proof:

- suppose $x = 0$, let $\epsilon > 0$, we have $x_n \to 0 \implies \exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $|x_n - 0| = |x_n| < \epsilon^2 \implies \forall n \geq M$, $|\sqrt{x_n} - \sqrt{x}| = |\sqrt{x_n}| < \sqrt{\epsilon^2} < \epsilon$
- suppose $x > 0$, we have

$$0 \leq |\sqrt{x_n} - \sqrt{x}| = \left|\frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}}\right|$$
$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}},$$

hence, $x_n \to x \implies |x_n - x| \to 0 \implies \frac{|x_n - x|}{\sqrt{x}} \to 0$
$\implies |\sqrt{x_n} - \sqrt{x}| \to 0$

Remark 3.26 Suppose the sequence $(x_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = x$. One can prove that

$$\lim_{n\to\infty} x_n^k = x^k$$

by induction. Moreover, if $x_n \geq 0$ for all $n \in \mathbf{N}$, one can also prove that

$$\lim_{n\to\infty} \sqrt[k]{x_n} = \sqrt[k]{x}.$$

Theorem 3.27 If $(x_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = x$, then $(|x_n|)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} |x_n| = |x|$.

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proof: let $\epsilon > 0$; $x_n \to x \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|x_n - x| < \epsilon$; hence, by reverse triangle inequality, for all $n \geq M$, we have

$$||x_n| - |x|| \leq |x_n - x| < \epsilon$$

Some special sequences

Theorem 3.28 If $p > 0$ then $\lim_{n\to\infty} n^{-p} = 0$.

proof: let $\epsilon > 0$, choose $M \in \mathbf{N}$ such that $M > (1/\epsilon)^{1/p}$, then for all $n \geq M$, $|n^{-p} - 0| = 1/n^p \leq 1/M^p < \epsilon$

Theorem 3.29 If $p > 0$ then $\lim_{n\to\infty} p^{1/n} = 1$.

proof:

- if $p = 1$, $\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} 1^{1/n} = 1$
- suppose $p > 1$
  - $p > 1 \implies p^{1/n} > 1^{1/n} = 1 \implies p^{1/n} - 1 > 0$

– according to the Bernoulli's inequality (example 1.8), we have

$$\left(1 + (p^{1/n} - 1)\right)^n \geq 1 + n(p^{1/n} - 1) \implies \frac{p-1}{n} \geq p^{1/n} - 1 > 0$$

– $\frac{p-1}{n} \to 0 \implies p^{1/n} - 1 \to 0 \implies p^{1/n} \to 1$

- if $0 < p < 1 \implies 1/p > 1$, hence, we have

$$\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} \frac{1}{(1/p)^{1/n}} = 1/1 = 1$$

Theorem 3.30 The sequence $\left(n^{1/n}\right)_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} n^{1/n} = 1$.

proof:

- one can simply show that $n^{1/n} \geq 1$ by induction $\implies n^{1/n} - 1 > 0$
- according to the binomial theorem, for all $x, y \in \mathbf{R}$ and $n \in \mathbf{N}$, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- let $x = 1$, $y = n^{1/n} - 1$, for all $n > 1$, we have

$$
\begin{aligned}
& n = \left(1 + n^{1/n} - 1\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(n^{1/n} - 1\right)^k \geq \binom{n}{2} \left(n^{1/n} - 1\right)^2 \\
\implies\quad & n \geq \frac{n!}{2!(n-2)!} \left(n^{1/n} - 1\right)^2 = \frac{1}{2} n(n-1) \left(n^{1/n} - 1\right)^2 \\
\implies\quad & \sqrt{\frac{2}{n-1}} \geq n^{1/n} - 1 > 0 \\
\implies\quad & n^{1/n} - 1 \to 0 \implies n^{1/n} \to 1
\end{aligned}
$$

Limit superior and limit inferior

Definition 3.31 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Define, if the limits exist, the **limit superior** and **limit inferior** respectively, as

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} (\sup\{x_k \mid k \geq n\})$$

and

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} (\inf\{x_k \mid k \geq n\}).$$

Theorem 3.32 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence; let

$$a_n = \sup\{x_k \mid k \geq n\} \quad \text{and} \quad b_n = \inf\{x_k \mid k \geq n\}.$$

Then:

- The sequence $(a_n)_{n=1}^{\infty}$ is monotone decreasing and bounded.
- The sequence $(b_n)_{n=1}^{\infty}$ is monotone increasing and bounded.
- We have $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

proof:

- we first prove the following lemma:

  Lemma 3.33 Let $A, B \subseteq \mathbf{R}$, $A, B \neq \emptyset$, and A, B are bounded. If $A \subseteq B$ then we have $\inf B \leq \inf A \leq \sup A \leq \sup B$.

  – $A \subseteq B \implies \sup B$ is an upper bound of $A \implies \sup A \leq \sup B$

– similarly, $\inf B$ is an lower bound of $A \implies \inf B \leq \inf A$

– $A, B \neq \emptyset \implies \inf A \leq \sup A \implies \inf B \leq \inf A \leq \sup A \leq \sup B$

- we now show the first two statements in the theorem
  – $(x_n)_{n=1}^{\infty}$ is bounded $\implies$ there exists some $B \geq 0$ such that $-B \leq x_n \leq B$
  – we have $\forall n \in \mathbf{N}$, $\{x_k \mid k \geq n+1\} \subseteq \{x_k \mid k \geq n\} \subseteq \{x_n \mid n \in \mathbf{N}\}$, according to lemma 3.33, this implies that

$$-B \leq b_n \leq b_{n+1} \leq a_{n+1} \leq a_n \leq B,$$

  i.e., $(a_n)_{n=1}^{\infty}$ is bounded monotone decreasing and $(b_n)_{n=1}^{\infty}$ is bounded monotone increasing ($\implies$ $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge)

- according to the previous inequalities, we have $b_n \leq a_n$ for all $n \in \mathbf{N}$ $\implies \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n$ (theorem 3.23), *i.e.*, $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

Example 3.34 We have the following:

$$\limsup_{n\to\infty} (-1)^n = 1 \quad \text{and} \quad \liminf_{n\to\infty} (-1)^n = -1.$$

proof: we have $\forall n \in \mathbf{N}$, the set $\{(-1)^k \mid k \geq n\} = \{-1, 1\} \implies$ $\sup\{(-1)^k \mid k \geq n\} = 1$ and $\inf\{(-1)^k \mid k \geq n\} = -1 \implies$ $\limsup_{n\to\infty} (-1)^n = 1$ and $\liminf_{n\to\infty} (-1)^n = -1$

Example 3.35 We have $\limsup_{n\to\infty} \frac{1}{n} = \liminf_{n\to\infty} \frac{1}{n} = 0$.

proof: $\forall n \in \mathbf{N}$, $\sup\{1/k \mid k \geq n\} = 1/k$ and $\inf\{1/k \mid k \geq n\} = 0$, hence,

$$\limsup_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{k} = 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} 0 = 0$$

Bolzano-Weierstrass theorem

Theorem 3.36 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Then, there exists subsequences $(x_{n_i})_{i=1}^{\infty}$ and $(x_{m_i})_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n \quad \text{and} \quad \lim_{i\to\infty} x_{m_i} = \liminf_{n\to\infty} x_n.$$

proof: let $a_n = \sup\{x_k \mid k \geq n\}$

- $a_1 = \sup\{x_k \mid k \geq 1\} \implies \exists n_1 \geq 1$ such that $a_1 - 1 < x_{n_1} \leq a_1$
- $a_{n_1+1} = \sup\{x_k \mid k \geq n_1 + 1\} \implies \exists n_2 > n_1$ such that $a_{n_1+1} - \frac{1}{2} < x_{n_2} \leq a_{n_1+1}$
- $a_{n_2+1} = \sup\{x_k \mid k \geq n_2 + 1\} \implies \exists n_3 > n_1$ such that $a_{n_2+1} - \frac{1}{3} < x_{n_3} \leq a_{n_2+1}$

- repeatedly, we can find a sequence of integers $n_1 < n_2 < \cdots$ such that

$$a_{n_{i-1}+1} - \frac{1}{i} < x_{n_i} \leq a_{n_{i-1}+1}$$

  (defining $n_0 = 0$)

- $\left(a_{n_{i-1}+1}\right)_{i=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$, and $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n \implies \lim_{n\to\infty} a_{n_{i-1}+1} = \limsup_{n\to\infty} x_n$ $\implies \lim_{n\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n$

- similarly, we can find a subsequence of $(x_n)_{n=1}^{\infty}$ that converges to $\liminf_{n\to\infty} x_n$

Theorem 3.37 *Bolzano-Weierstrass.* Every bounded sequence consisting of real numbers has a convergent subsequence.

Theorem 3.38 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Then, $(x_n)_{n=1}^{\infty}$ converges if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

proof:

- suppose $\lim_{n\to\infty} x_n = x$, then the subsequences that converge to $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ must converge to x (theorem 3.18)

- suppose $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$, for all $n \in \mathbf{N}$, according to the squeeze theorem,

$$\inf\{x_k \mid k \geq n\} \leq x_n \leq \sup\{x_k \mid k \geq n\} \implies \lim_{n\to\infty} x_n = x$$

Cauchy sequences

Definition 3.39 A sequence $(x_n)_{n=1}^{\infty}$ is **Cauchy** if for all $\epsilon > 0$, there exists an $M \in \mathbf{N}$ such that for all $n, k \geq M$, we have $|x_n - x_k| < \epsilon$.

Remark 3.40 A sequence $(x_n)_{n=1}^{\infty}$ is not Cauchy if there exists some $\epsilon > 0$, such that for all $M \in \mathbf{N}$, there exists some $n, k \geq M$, so that we have $|x_n - x_k| \geq \epsilon$.

Example 3.41 The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is Cauchy.

proof: let $\epsilon > 0$, choose $M \in \mathbf{N}$ such that $M > 2/\epsilon$, then for all $n, k \geq M$, we have

$$\left|\frac{1}{n} - \frac{1}{k}\right| \leq \frac{1}{n} + \frac{1}{k} \leq \frac{2}{M} < \epsilon$$

Example 3.42 The sequence $((-1)^n)_{n=1}^{\infty}$ is not Cauchy.

proof: let $\epsilon = 1$, $M \in \mathbf{N}$, $n = M$, $k = M + 1$, then

$$\left|(-1)^n - (-1)^k\right| = 2 \geq \epsilon$$

Theorem 3.43 If the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy, then it is bounded.

proof:

- let $\epsilon = 1$, $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies \exists M \in \mathbf{N}$ such that $\forall n, k \geq M$, $|x_n - x_k| < 1$
- let $k = M \implies \forall n \geq M$, $|x_n - x_M| < 1 \implies \forall n \geq M$, we have $|x_n| < |x_M| + 1$
- take $B = \max\{|x_1|, |x_2|, \ldots, |x_M|, |x_M| + 1\}$, then $|x_n| \leq B$ for all $n \in \mathbf{N}$

Theorem 3.44 If the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy and a subsequence $(x_{n_i})_{i=1}^{\infty}$ converges, then $(x_n)_{n=1}^{\infty}$ converges.

proof: let $\epsilon > 0$

- $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies \exists M_1 \in \mathbf{N}$ such that $\forall n, k \geq M_1$, $|x_n - x_k| < \epsilon/2$
- let $\lim_{i\to\infty} x_{n_i} = x \implies \exists M_2 \in \mathbf{N}$ such that $\forall i \geq M_2$, $|x_{n_i} - x| < \epsilon/2$
- let $M = \max\{M_1, M_2\}$, then $\forall k \geq M$, $n_k \geq k \geq M_1$, $n_k \geq k \geq M_2$, hence,

$$|x_k - x| \leq |x_k - x_{n_k}| + |x_{n_k} - x| < \epsilon/2 + \epsilon/2 = \epsilon$$

Theorem 3.45 *Completeness of the real numbers.* A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is Cauchy if and only if the sequence $(x_n)_{n=1}^{\infty}$ is convergent.

proof:

- suppose $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies$ $(x_n)_{n=1}^{\infty}$ is bounded (theorem 3.43)
$\implies$ by theorem 3.37 there exists convergent subsequence of $(x_n)_{n=1}^{\infty}$
$\implies$ $(x_n)_{n=1}^{\infty}$ is convergent (theorem 3.44)

- suppose $\lim_{n\to\infty} x_n = x$, let $\epsilon > 0$, then $\exists M \in \mathbf{N}$, $\forall n \geq M$, we have $|x_n - x| < \epsilon/2$; let $k \geq M$, then

$$|x_n - x_k| \leq |x_n - x| + |x - x_k| < \epsilon/2 + \epsilon/2 = \epsilon$$

Remark 3.46 We say a set is **Cauchy-complete**, or just **complete**, if all Cauchy sequence of elements in the set converges to some point in the set. Theorem 3.45 indicates that $\mathbf{R}$ is complete.

Remark 3.47 The set $\mathbf{Q}$ is *not* complete. Since $\mathbf{Q}$ does not have the least upper bound property, then, $e.g.$, $\sup\{x_n \mid n \in \mathbf{N}\}$, $\sup\{x_k \mid k \geq n\}$, $etc.$, might not exist in $\mathbf{Q}$.