3. Disciplined convex analysis and programming

- methods of convexity verification
- restriction of a function to a line
- second-order conditions
- constructive convex analysis
- disciplined convex programming

Methods of convexity verification

practical methods for establishing convexity of a set $C \subseteq \mathbf{R}^n$

1. apply definition

$$x_1, x_2 \in C, \quad \theta \in [0,1] \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

2. show that C is obtained from simple convex sets, *e.g.*,

   - hyperplanes and halfspaces
   - polyhedra and simplexes
   - balls and ellipsoids
   - . . .

   by operations that preserve convexity, *e.g.*,

   - intersection
   - Cartesian product
   - affine transformation
   - . . .

for establishing convexity of a function $f : \mathbf{R}^n \to \mathbf{R}$

1. apply definition (Jensen's inequality)

$$x_1, x_2 \in \mathbf{R}^n, \quad \theta \in [0,1] \implies f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$

2. prove second-order condition (for twice differentiable f)

$$\nabla^2 f(x) \succeq 0 \text{ for all } x \in \mathbf{R}^n$$

3. show that f is obtained from simple convex functions by operations that preserve convexity, *e.g.*,
   - nonnegative weighted sum
   - pointwise maximum and supremum
   - composition with affine function
   - general composition
   - . . .

Restriction of a function to a line

show Jensen's inequality for $f\colon \mathbf{R}^n \to \mathbf{R}$ is equivalent to proving

$$g(t) = f(x+tv) \text{ convex (in } t\text{) for all } x \in \mathbf{dom}\, f,\ v \in \mathbf{R}^n$$

on its domain $\{t \in \mathbf{R} \mid x + tv \in \mathbf{dom}\, f\}$

i.e., f convex $\iff$ f convex along all lines in its domain

example: for $f\colon \mathbf{S}^n \to \mathbf{R}$ with $f(X) = \log\det X$, $\mathbf{dom}\, f = \mathbf{S}^n_+$,

$$\begin{aligned} g(t) &= \log\det(Z+tV) \\ &= \log\det Z + \log\det(I + tZ^{-1/2}VZ^{-1/2}) \\ &= \log\det Z + \textstyle\sum_{i=1}^n \log(1+t\lambda_i) \end{aligned}$$

where λ_i are the eigenvalues of $Z^{-1/2}VZ^{-1/2}$

g concave in t for all $Z \succ 0$, V $\implies$ f concave

Second-order conditions

consider log-sum-exp function $f : \mathbf{R}^n \to \mathbf{R}$ given by

$$f(x) = \log \sum_{i=1}^{n} \exp x_i, \quad \mathbf{dom}\, f = \mathbf{R}^n$$

we have

$$\nabla^2 f(x) = \frac{1}{\left(\mathbf{1}^T z\right)^2} \left((\mathbf{1}^T z)\, \mathbf{diag}(z) - z z^T \right)$$

where $z = (e^{x_1}, \ldots, e^{x_n})$

to show $\nabla^2 f(x) \succeq 0$, we need to show for all $v \in \mathbf{R}^n$,

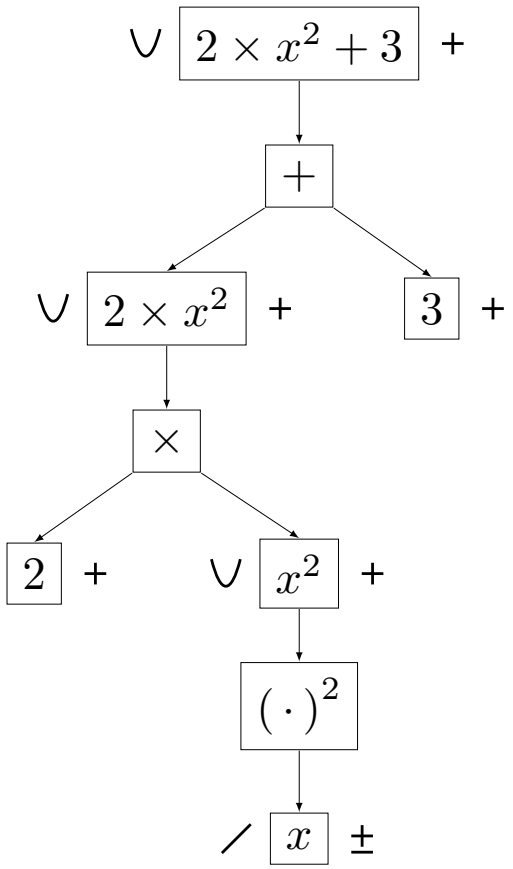
$$v^T \nabla^2 f(x) v = \frac{1}{\left(\mathbf{1}^T z\right)^2} \left(\left(\sum_{i=1}^{n} z_i \right) \left(\sum_{i=1}^{n} z_i v_i^2 \right) - \left(\sum_{i=1}^{n} z_i v_i \right)^2 \right) \geq 0$$

which holds by Cauchy-Schwarz inequality

Constructive convex analysis

basic ideas:

- parse target function f into a *composition tree*
- nodes consist of
  - atomic functions with known convexity, or
  - convexity preserving operations
- traverse the tree from leaves to root, check whether convexity is preserved



example: $f(x) = 2x^2 + 3$

Disciplined convex programming (DCP)

a DCP problem has the form

$$\begin{array}{ll} \text{minimize/maximize} & f(x) \\ \text{subject to} & p_i(x) \sim q_i(x), \quad i = 1, \ldots, m \end{array}$$

- variable $x \in \mathbf{R}^n$, objective $f\colon \mathbf{R}^n \to \mathbf{R}$, constraints $p_i, q_i\colon \mathbf{R}^n \to \mathbf{R}$
- relational operator $\sim$ is one of $=$, $\leq$, and $\geq$

convexity follows from the following rules:

- f convex if minimization, concave if maximization
- p_i convex, q_i concave if $\sim$ is $\leq$
- p_i concave, q_i convex if $\sim$ is $\geq$
- p_i and q_i affine if $\sim$ is $=$

example: DCP problem with variables $x \in \mathbf{R}_+$, $y \in \mathbf{R}$

$$
\begin{array}{ll}
\text{minimize} & cy \\
\text{subject to} & \exp(y) \leq \log(a\sqrt{x} + b) \\
& ax + by = d,
\end{array}
$$

