Lecture Notes for Real Analysis

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1. Basic set theory

• sets

- mathematical induction
- functions
- cardinality

Basic set theory

Sets

Definition 1.1 A set is a collection of objects called elements or members. A set with no objects is called the **empty set** and is denoted by \emptyset (or sometimes by $\{\}$).

notation:

- $a \in S$ means that 'a is an element in S'
- $a \notin S$ means that 'a is not an element in S'
- ∀ means 'for all'
- \exists means 'there exists'
- \exists ! means 'there exists a unique'
- \implies means 'implies'
- \iff means 'if and only if'

Definition 1.2

- A set A is a subset of a set B if $x \in A$ implies $x \in B$, denoted as $A \subseteq B$.
- Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$, denoted as A = B.
- A set A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$, denoted as $A \subsetneq B$.

set building notation: we write

$$\{x \in A \mid P(x)\} \quad \text{or} \quad \{x \mid P(x)\}$$

to mean 'all $x \in A$ that satisfies property P(x)'

examples:

- $\mathbf{N} = \{1, 2, 3, 4, \ldots\}$: the set of natural numbers
- $\mathbf{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$: the set of integers
- $\mathbf{Q} = \{m/n \mid m, n \in \mathbf{Z}, n \neq 0\}$: the set of rational numbers
- $\bullet~\mathbf{R}:$ the set of real numbers

it follows that $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$

Basic set theory

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Definition 1.3 Given sets A and B:

- The union of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- The intersection of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
- The set difference of A and B is the set $A \setminus B = \{x \in A \mid x \notin B\}$.
- The complement of A is the set $A^c = \{x \mid x \notin A\}$.
- A and B are **disjoint** if $A \cap B = \emptyset$.

Theorem 1.4 De Morgan's Laws. If A, B, C are sets, then

- $(B \cup C)^c = B^c \cap C^c$;
- $(B \cap C)^c = B^c \cup C^c;$
- $A \setminus (B \cup C) = A \setminus B \cap A \setminus C$;
- $A \setminus (B \cap C) = A \setminus B \cup A \setminus C.$

we prove the first statement:

• let B, C be sets, we need to show that

 $(B \cup C)^c \subseteq B^c \cap C^c$ and $B^c \cap C^c \subseteq (B \cup C)^c$

- $x \in (B \cup C)^c \implies x \notin B \cup C \implies x \notin B$ and $x \notin C$ $\implies x \in B^c$ and $x \in C^c \implies x \in B^c \cap C^c \implies (B \cup C)^c \subseteq B^c \cap C^c$
- $x \in B^c \cap C^c \implies x \in B^c \text{ and } x \in C^c \implies x \notin B \text{ and } x \notin C$ $\implies x \notin B \cup C \implies x \in (B \cup C)^c \implies B^c \cap C^c \subseteq (B \cup C)^c$

Basic set theory

Mathematical induction

Axiom 1.5 Well ordering property. If the set $S \subseteq \mathbb{N}$ is nonempty, then there exists some $x \in S$ such that $x \leq y$ for all $y \in S$, *i.e.*, the set S always has a **least element**.

Theorem 1.6 Induction. Let P(n) be a statement depending on $n \in \mathbb{N}$. Assume that we have:

- 1. *Base case.* The statement P(1) is true.
- 2. Inductive step. If P(m) is true then P(m+1) is true.

Then, P(n) is true for all $n \in \mathbf{N}$.

proof:

- suppose $S \neq \emptyset$, then S has a least element $m \in S$
- since P(1) is true, we have $m \neq 1$, *i.e.*, m > 1
- since m is a least element, we have $m-1 \notin S \implies P(m-1)$ is true
- this implies that P(m) is true $\implies m \notin S$, which is a contradiction
- hence, $S = \emptyset$, *i.e.*, P(n) is true for all $n \in \mathbb{N}$

Example 1.7 For all $c \in \mathbf{R}$, $c \neq 1$, and for all $n \in \mathbf{N}$,

$$1 + c + c^{2} + \dots + c^{n} = \frac{1 - c^{n+1}}{1 - c}$$

proof:

- the base case (n = 1): the left hand side of the equation is 1 + c; the right hand side is $\frac{1-c^2}{1-c} = \frac{(1+c)(1-c)}{1-c} = 1 + c$, which equals to the left hand side
- the inductive step: assume that the equation is true for $k \in \mathbf{N}$, *i.e.*,

$$1 + c + c^{2} + \dots + c^{k} = \frac{1 - c^{k+1}}{1 - c},$$

we have

$$1 + c + c^{2} + \dots + c^{k} + c^{k+1} = \frac{1 - c^{k+1}}{1 - c} + c^{k+1}$$
$$= \frac{1 - c^{k+1} + c^{k+1} - c^{(k+1)+1}}{1 - c} = \frac{1 - c^{(k+1)+1}}{1 - c}$$

Basic set theory

Example 1.8 Bernoulli's inequality. For all $c \ge -1$, $(1+c)^n \ge 1 + nc$ for all $n \in \mathbf{N}$.

proof:

- for the base case (n=1), we have $(1+c)^1 \ge 1+1 \cdot c$
- the inductive step: suppose $m \in \mathbf{N}$, m > 1 and $(1+c)^m \ge 1+mc$, then

$$(1+c)^{m+1} \ge (1+mc)(1+c) = 1 + (m+1)c + mc^2 \ge 1 + (m+1)c$$

Functions

Definition 1.9 If A and B are sets, a function $f: A \to B$ is a mapping that assigns each $x \in A$ to a unique element in B denoted f(x).

Definition 1.10 Consider a function $f: A \to B$. Define the **image** (or direct image) of a subset $C \subseteq A$ as

 $f(C) = \{ f(x) \in B \mid x \in C \}.$

Define the **inverse image** of a subset $D \subseteq B$ as

$$f^{-1}(D) = \{ x \in A \mid f(x) \in D \}.$$

examples:

- $f: \{1, 2, 3, 4\} \rightarrow \{a, b\}$ where f(1) = f(2) = a, f(3) = f(4) = b, we have $f(\{1, 2\}) = \{a\}, f^{-1}(\{b\}) = \{3, 4\}$
- $f: \mathbf{R} \to \mathbf{R}$ where $f(x) = \sin(\pi x)$, we have f([0, 1/2]) = [0, 1], $f^{-1}(\{0\}) = \mathbf{Z}$

Basic set theory

Definition 1.11 Let $f: A \to B$ be a function.

- The function f is **injective** or **one-to-one** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- The function f is surjective or onto if f(A) = B.
- The function f is bijective if f is both surjective and injective. In this case, the function f⁻¹: B → A is the inverse function of f, which assigns each y ∈ B to the unique x ∈ A such that f(x) = y.
- if the function f is a bijection, then $f(f^{-1}(x)) = x$
- example: for the bijection $f: \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$, we have $f^{-1}(x) = \sqrt[3]{x}$

Definition 1.12 Consider $f: A \to B$ and $g: B \to C$. The **composition** of the functions f and g is the function $g \circ f: A \to C$ defined as

$$(g \circ f)(x) = g(f(x)).$$

• example: if $f(x) = x^3$ and $g(y) = \sin(y)$, then $(g \circ f)(x) = \sin(x^3)$

Cardinality

Definition 1.13 We state that the two sets A and B have the same cardinality if there exists a bijection $f: A \rightarrow B$.

notation:

- |A| denotes the cardinality of the set A
- |A| = |B| if the sets A and B have the same cardinality
- |A| = n if $|A| = |\{1, \dots, n\}|$
- $|A| \leq |B|$ if there exists an injection $f: A \to B$
- |A| < |B| if $|A| \le |B|$ and $|A| \ne |B|$

Basic set theory

Theorem 1.14

- If |A| = |B|, then |B| = |A|.
- If |A| = |B|, and |B| = |C|, then |A| = |C|.

proof:

- show that the inverse function $f^{-1} \colon B \to A$ of $f \colon A \to B$ is a bijection
- show that the composition $g\circ f\colon A\to C$ of functions $f\colon A\to B$ and $g\colon B\to C$ is a bijection

Theorem 1.15 Cantor-Schröder-Bernstein. If $|A| \leq |B|$ and $|B| \leq |A|$ then |A| = |B|.

Definition 1.16 The set A is countably finite if $|A| = |\mathbf{N}|$. Specifically, the set A is finite if $|A| = n \in \mathbf{N}$. The set A is countable if A is finite or countably infinite. Otherwise, we say A is uncountable.

Example 1.17 The set of even natural numbers and the set of odd natural numbers have the same cardinality as N, *i.e.*, $|\{2n \mid n \in \mathbf{N}\}| = |\{2n - 1 \mid n \in \mathbf{N}\}| = |\mathbf{N}|$.

proof: consider the bijection $f: \mathbf{N} \to \{2n \mid n \in \mathbf{N}\}$ given by f(n) = 2n and $g: \mathbf{N} \to \{2n-1 \mid n \in \mathbf{N}\}$ given by g(n) = 2n-1

Example 1.18 The set of all integers has the same cardinality as N, *i.e.*, $|\mathbf{Z}| = |\mathbf{N}|$.

proof: consider the bijection $f: \mathbf{Z} \to \mathbf{N}$ given by

$$f(n) = \begin{cases} 2n & n \ge 0\\ -(2n+1) & n < 0 \end{cases}$$

Basic set theory

Definition 1.19 The **powerset** of a set A, denoted by $\mathcal{P}(A)$, is the set of all subsets of A, *i.e.*, $\mathcal{P}(A) = \{B \mid B \subseteq A\}$.

• for a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n

examples:

- $A = \emptyset$ then $\mathcal{P}(A) = \{\emptyset\}$
- $A = \{1\}$ then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$
- $A = \{1, 2\}$ then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Theorem 1.20 Cantor. If A is a set, then $|A| < |\mathcal{P}(A)|$.

• therefore, $|\mathbf{N}| < |\mathcal{P}(\mathbf{N})| < |\mathcal{P}(\mathcal{P}(\mathbf{N}))| < \cdots$, *i.e.*, there are infinite number of infinite sets

proof:

we first show that $|A| \leq |\mathcal{P}(A)|$

- consider the function $f: A \to \mathcal{P}(A)$ given by $f(x) = \{x\}$
- the function f is a injection since

$$f(x_1) = f(x_2) \implies \{x_1\} = \{x_2\} \implies x_1 = x_2$$

we now show that $|A| \neq |\mathcal{P}(A)|$ by contradiction

- suppose $|A| = |\mathcal{P}(A)|$, then there is a surjection $g \colon A \to \mathcal{P}(A)$
- consider the set $B \subseteq A$ given by

$$B = \{x \in A \mid x \notin g(x)\} \in \mathcal{P}(A)$$

- since g is surjective and $B \in \mathcal{P}(A)$, there exists a $b \in A$ such that g(b) = B
- there are two cases
 - $1. \ b \in B \implies b \notin g(b) \implies b \notin B$

2. $b \notin B \implies b \notin g(b) \implies b \in B$

where in either case we obtain a contradiction

• hence, $g \text{ is not surjective } \Longrightarrow |A| \neq |\mathcal{P}(A)|$

Corollary 1.21 For all $n \in \mathbb{N} \cup \{0\}$, $n < 2^n$.

Basic set theory

2. Real numbers

- ordered sets
- least upper bound property
- fields
- real numbers
- archimedian property
- using supremum and infimum
- absolute value
- triangle inequality
- uncountabality of the real numbers

Real numbers

2-1

Ordered sets

Definition 2.1 An ordered set is a set S with a relation < called an 'ordering' such that:

- 1. Trichotomy. For all $x, y \in S$, either x < y, x = y, or x > y.
- 2. Transitivity. If $x, y, z \in S$ have x < y and y < z, then x < z.

examples:

- Z is an ordered set with ordering $m > n \Longleftrightarrow m n \in \mathbf{N}$
- Q is an ordered set with ordering $p > q \iff p q = m/n$ for some $m, n \in \mathbf{N}$
- $\mathbf{Q} \times \mathbf{Q}$ is an ordered set with dictionary ordering $(q,r) > (s,t) \iff q > s$, or q = s and r > t
- the set $\mathcal{P}(\mathbf{N})$ with ordering defined by $A \prec B$ if $A \subseteq B$ is *not* an ordered set

Definition 2.2 Let S be an ordered set and let $E \subseteq S$, then:

- If there exists some b ∈ S such that x ≤ b for all x ∈ E, then E is bounded above and b is an upper bound of E.
- If there exists some c ∈ S such that x ≥ c for all x ∈ E, then E is bounded below and c is a lower bound of E.
- If there exists an upper bound b₀ of E such that b₀ ≤ b for all upper bounds b of E, then b₀ is the least upper bound or the supremum of E, written as

$$b_0 = \sup E.$$

• If there exists a lower bound c_0 of E such that $c_0 \ge c$ for all lower bounds c of E, then c_0 is the **greatest lower bound** or the **infimum** of E, written as

 $c_0 = \inf E.$

Real numbers

examples:

- $S = \mathbf{Z}$ and $E = \{-2, -1, 0, 1, 2\}$, then $\inf E = -2$ and $\sup E = 2$
- $S = \mathbf{Q}$ and $E = \{q \in \mathbf{Q} \mid 0 \le q < 1\}$, then $\inf E = 0$ and $\sup E = 1 \notin E$, *i.e.*, the supremum or infimum need not be in E
- $S = \mathbf{Z}$ and $E = \mathbf{N}$, then $\inf E = 1$ but $\sup E$ does not exist

Definition 2.3 Least upper bound property. An ordered set S has the least upper bound property if every $E \subseteq S$ which is nonempty and bounded above has a supremum in S.

example: $-\mathbf{N} = \{-1, -2, -3, ...\}$, to show this (informally), suppose $E \subseteq -\mathbf{N}$ is bounded above, then $-E \subseteq \mathbf{N}$ is bounded below and according to the well ordering principle, -E has a least element $x \in -E$, and thus $-x = \sup E$

Theorem 2.4 If $x \in \mathbf{Q}$ and

$$x = \sup\{q \in \mathbf{Q} \mid q > 0, q^2 < 2\},\$$

then $x \ge 1$ and $x^2 = 2$.

proof: let $E = \{q \in \mathbf{Q} \mid q > 0, q^2 < 2\}$ • $x \ge 1$ since $1 \in E \implies \sup E \ge 1$ • we show $x^2 \ge 2$ by contradiction: suppose $x^2 < 2$, let $h = \min\{\frac{1}{2}, \frac{2-x^2}{2(2x+1)}\}$ - since $x \ge 1$ and $x^2 < 2$, we have $0 < h \le 1/2 < 1$ - $h < 1 \implies (x+h)^2 = x^2 + 2hx + h^2 < x^2 + 2hx + h$ - since $h \le \frac{2-x^2}{2(2x+1)}$, we have $(x+h)^2 < x^2 + (2x+1)h \le x^2 + \frac{1}{2}(2-x^2) < x^2 + 2 - x^2 = 2 \implies x+h \in E$ - $h > 0 \implies x+h > x$, but $x+h \in E \implies x$ is not an upper bound for E, *i.e.*, $x \ne \sup E$, which is a contradiction

Real numbers

we now show x² ≥ 2 by contradiction: suppose x² > 2, let h = x²-2/2x
since x² > 2 and x ≥ 1, we have h > 0
h > 0 ⇒ (x - h)² = x² - 2hx + h² > x² - 2hx = x² - (x² - 2) = 2
let q ∈ E, then q² < 2 < (x - h)², hence
(x - h)² - q² = ((x - h) + q)((x - h) - q) > 0 ⇒ (x - h) - q > 0, *i.e.*, x - h > q for all q ∈ E ⇒ x - h is an upper bound for E
h > 0 ⇒ x > x - h ⇒ x ≠ sup E, which is a contradiction
therefore, x² = 2

Theorem 2.5 The set $E = \{q \in \mathbf{Q} \mid q > 0, q^2 < 2\}$ does not have a supremum in \mathbf{Q} .

proof (by contradiction): suppose there exists some $x \in \mathbf{Q}$ such that $x = \sup E$

- by theorem 2.4, we have $x \ge 1$ and $x^2 = 2$
- in particular, x > 1 since if $x = 1 \implies x^2 = 1 \neq 2$
- $x \in \mathbf{Q} \implies$ there exist $m, n \in \mathbf{N}$ (m > n) such that x = m/n, *i.e.*, $m = nx \in \mathbf{N}$
- let $S = \{k \in \mathbf{N} \mid kx \in \mathbf{N}\} \subseteq \mathbf{N}$, then $S \neq \emptyset$ since $n \in S$
- by the well ordering property, there is a least element $k_0 \in S$
- let $k_1 = k_0(x-1) = k_0x k_0 \in \mathbb{Z}$, in particular, $k_1 \in \mathbb{N}$ since $x > 1 \implies k_1 > 0$
- $x^2 = 2 \implies x < 2$ as otherwise $x^2 \ge 4$, hence

$$k_1 = k_0(x-1) < k_0(2-1) = k_0 \implies k_1 \notin S$$

• $k_1 = k_0(x-1) \implies k_1x = k_0x^2 - k_0x$, since $x^2 = 2$, we have

$$k_1 x = 2k_0 - k_0 x = k_0 - k_0 (x - 1) = k_0 - k_1 \in \mathbf{N} \implies k_1 \in S$$

which is a contradiction

Real numbers

Fields

Definition 2.6 A set F is a **field** if it has two operations: addition (+) and multiplication (\cdot) with the following properties.

- (A1) If $x, y \in F$ then $x + y \in F$.
- (A2) Commutativity. For all $x, y \in F$, x + y = y + x.
- (A3) Associativity. For all $x, y, z \in F$, (x + y) + z = x + (y + z).
- (A4) There exists an element $0 \in F$ such that 0 + x = x = x + 0 for all $x \in F$.
- (A5) For all $x \in F$, there exists a $y \in F$ such that x + y = 0, denoted by y = -x.
- (M1) If $x, y \in F$ then $x \cdot y \in F$.
- (M2) Commutativity. For all $x, y \in F$, $x \cdot y = y \cdot x$.
- (M3) Associativity. For all $x, y, z \in F$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- (M4) There exists an element $1 \in F$ such that $1 \cdot x = x = x \cdot 1$ for all $x \in F$.
- (M5) For all $x \in F \setminus \{0\}$, there exists an $x^{-1} \in F$ such that $x \cdot x^{-1} = 1$.
 - (D) Distributativity. For all $x, y, z \in F$, $(x + y) \cdot z = x \cdot z + y \cdot z$.

examples:

- $\bullet~{\bf Q}$ is a field
- Z is not a field since it fails (M5)
- $\mathbf{Z}_2 = \{0, 1\}$ where $1 + 1 = 0 \pmod{2}$ is a field
- $\mathbf{Z}_3 = \{0, 1, 2\}$ with $c = a + b \pmod{3}$, *i.e.*,

2+1=3=0 and $2 \cdot 2 = 4 = 3+1 = 1$,

is a field

Theorem 2.7 If $x \in F$ where F is a field then 0x = 0.

proof: $xx = (x+0)x = xx + 0x \implies 0x = 0$

Real numbers

2-9

Definition 2.8 A field F is an **ordered field** if F is also an ordered set with ordering < and satisfies:

- 1. For all $x, y, z \in F$, $x < y \implies x + z < y + z$.
- 2. If x > 0 and y > 0 then xy > 0.

If x > 0 we say x is **positive**, and if $x \ge 0$ we say x is **nonnegative**.

examples:

- Q is an ordered field
- $\mathbf{Z}_2 = \{0, 1\}$ where 1 + 1 = 0 is not a ordered field (if $0 > 1 \implies 0+1 > 1+1 \implies 1 > 0$; if $1 > 0 \implies 1+1 > 1+0 \implies 0 > 1$)

Theorem 2.9 Let F be an ordered field and $x, y, z, w \in F$, then:

- If x > 0 then -x < 0 (and vice versa).
- If x > 0 and y < z then xy < xz.
- If x < 0 and y < z then xy > xz.
- If $x \neq 0$ then $x^2 > 0$.
- If 0 < x < y then 0 < 1/y < 1/x.
- If 0 < x < y then $x^2 < y^2$.
- If $x \leq y$ and $z \leq w$ then $x + z \leq y + w$.

Theorem 2.10 Let $x, y \in F$ where F is an ordered field. If x > 0 and y < 0 or x < 0 and y > 0, then xy < 0.

proof:

- $\bullet \ x>0, \ y<0 \implies x>0, \ -y>0 \implies -xy>0 \implies xy<0$
- $x < 0, y > 0 \implies -x > 0, y > 0 \implies -xy > 0 \implies xy < 0$

Real numbers

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Theorem 2.11 Greatest lower bound. Let F be an ordered field with the least upper bound property. If $A \subseteq F$ is nonempty and bounded below, then $\inf A$ exists in F.

proof: let $B = \{-x \mid x \in A\}$

- $A \subseteq F$ bounded below $\implies \exists a \in F, \forall x \in A, a \leq x \implies \exists a \in F, \forall x \in A, -a \geq -x \implies \exists a \in F, \forall x \in B, -a \geq x \implies B \subseteq F$ has an upper bound -a (this also shows that if a is a lower bound of A then -a is an upper bound of B)
- F has the least upper bound property $\implies \sup B \in F$
- let $c = \sup B$, then $c \ge x$, $\forall x \in B \implies -c \le -x$, $\forall x \in B \implies -c \le x$, $\forall x \in A \implies -c \in F$ is an lower bound of A
- we also have $c \leq -a$ with a being a lower bound of $A \implies -c \geq a \implies -c \in F$ is the greatest lower bound of A, *i.e.*, $-c = \inf A \in F$

Theorem 2.12 There exists a "unique" ordered field, labeled \mathbf{R} , such that $\mathbf{Q} \subseteq \mathbf{R}$ and \mathbf{R} has the least upper bound property.

 \bullet one can construct ${\bf R}$ using Dedekind cuts or as equivalence classes of Cauchy sequences.

Theorem 2.13 There exists a unique $r \in \mathbf{R}$ such that $r \ge 1$ and $r^2 = 2$, *i.e.*, $\sqrt{2} \in \mathbf{R}$ but $\sqrt{2} \notin \mathbf{Q}$.

proof: let $E = \{x \in \mathbf{R} \mid x > 0, x^2 < 2\} \subseteq \mathbf{R}$

- we have x < 2 for all x ∈ E (since if x ≥ 2 ⇒ x² ≥ 4) ⇒ E is bounded above ⇒ sup E exists in R
- let $r = \sup E$, using the same proof for theorem 2.4 we have $r \ge 1$ and $r^2 = 2$
- to show the uniqueness, suppose $\tilde{r} \ge 1$, $\tilde{r}^2 = 2$, then

$$r^2 - \tilde{r}^2 = 0 \implies (r + \tilde{r})(r - \tilde{r}) = 0 \implies r - \tilde{r} = 0 \implies r = \tilde{r}$$

(since $r \ge 1$, $\tilde{r} \ge 1 \implies r + \tilde{r} > 0$)

Real numbers

Theorem 2.14 If $x \in \mathbf{R}$ satisfies $x < \epsilon$ for all $\epsilon \in \mathbf{R}$, $\epsilon > 0$, then $x \leq 0$.

proof by contradiction:

- suppose x > 0 satisfies $x \le \epsilon$ for all $\epsilon > 0$
- $x > 0 \implies 2x > x > 0 \implies x > x/2 > 0$
- take $\epsilon = x/2$ we have $x > \epsilon > 0$, which is a contradiction

Theorem 2.15 Archimedian property. If $x, y \in \mathbf{R}$ and x > 0, then there exists an $n \in \mathbf{N}$ such that nx > y.

proof by contradiction:

- suppose $nx \leq y$ for all $n \in \mathbb{N} \implies \forall n \in \mathbb{N}$, $n \leq y/x \implies \mathbb{N}$ is bounded above by $y/x \implies$ there exists $\sup \mathbb{N} \in \mathbb{R}$
- let $a = \sup \mathbf{N} \implies a 1 < a$ is not an upper bound of $\mathbf{N} \implies \exists m \in \mathbf{N}$, $a - 1 < m \implies a < m + 1 \in \mathbf{N} \implies a$ is not an upper bound of \mathbf{N} , which is a contradiction

Theorem 2.16 Density of Q. If $x, y \in \mathbf{R}$ and x < y then there exists some $r \in \mathbf{Q}$ such that x < r < y.

proof:

• first suppose $0 \le x < y$, by the Archimedian property, we have

$$n(y-x) > 1 \implies ny > nx+1$$

for some $n \in \mathbf{N}$

Real numbers

2-15

- let $S = \{k \in \mathbb{N} \mid k > nx\} \subseteq \mathbb{N}$, by Archimedian property, there exists some $p \in \mathbb{N}$ such that $p > nx \implies S \neq \emptyset$
- by the well ordering property, there is a least element $m \in S$ such that m > nx
- $m \in \mathbf{N} \implies m \ge 1$
- if m = 1, then $m 1 = 0 \implies nx \ge m 1 = 0$ since $x \ge 0$
- if m > 1, then $m 1 \in \mathbb{N}$ but $m 1 \notin S$ since m > m 1 is the least element $\implies nx \ge m - 1 \implies m \le nx + 1 < ny$
- hence, we have

$$nx < m < ny \implies x < m/n < y$$

for some $m, n \in \mathbf{N}$, *i.e.*, there exists an $r = m/n \in \mathbf{Q}$ such that x < r < y

• now suppose x < 0, if x < 0 < y then simply take r = 0; if $x < y \le 0$, we have $0 \le -y < -x$, thus there exists some $\tilde{r} \in \mathbf{Q}$ such that

$$-y < \tilde{r} < -x \implies x < -\tilde{r} < y$$

(by the first case), *i.e.*, we have x < r < y by taking $r = -\tilde{r}$

Real numbers

Theorem 2.17 Suppose $S \subseteq \mathbf{R}$ is nonempty and bounded above. Then, $x = \sup S$ if and only if:

- 1. x is an upper bound of S.
- 2. For all $\epsilon > 0$, there exists some $y \in S$ such that $x \epsilon < y \le x$.

proof:

- first suppose $x = \sup S$
 - obviously, x is an upper bound of S
 - for all $\epsilon > 0$, we have $x > x \epsilon \implies x \epsilon$ is not an upper bound of S, *i.e.*, there exists some $y \in S$ such that $x \epsilon < y \le x$
- now suppose x is an upper bound of S, and satisfies $x \epsilon < y \le x$ for all $\epsilon > 0$ and for some $y \in S$, we only need to show that for all z that is an upper bound of S, we have $x \le z$
 - assume there exists an upper bound z of S smaller than $x, \ i.e., \ y \leq z < x$ for all $y \in S$
 - take $\epsilon = x z > 0$ (since x > z) $\implies x \ge y > x \epsilon = x x + z = z \implies y > z$ for some $y \in S$, *i.e.*, z is not an upper bound of S, which is a contradiction

Real numbers

Theorem 2.18 Let $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$, then $\sup S = 1$.

proof:

- if $n \in \mathbf{N}$, then $1 \frac{1}{n} < 1 \implies 1$ is an upper bound of S
- let $\epsilon > 0$, then by the Archimedian property, for some $n \in \mathbf{N}$, we have

$$n\epsilon > 1 \implies \epsilon > \frac{1}{n} \implies -\epsilon < -\frac{1}{n} \implies 1-\epsilon < 1-\frac{1}{n} \le 1$$

by theorem 2.17, we have $\sup S = 1$

Remark 2.19 We have similar property as theorem 2.17 for infimum. Suppose $S \subseteq \mathbf{R}$ is nonempty and bounded below, then $x = \inf S$ if and only if:

- x is a lower bound of S.
- For all $\epsilon > 0$, there exists some $y \in S$ such that $x \leq y < x + \epsilon$.

Using supremum and infimum

Definition 2.20 For $x \in \mathbf{R}$ and $A \subseteq \mathbf{R}$, define

 $x + A = \{x + a \mid a \in A\}, \qquad xA = \{xa \mid a \in A\}.$

Theorem 2.21 Let $A \subseteq \mathbf{R}$ be nonempty, we have:

- If $x \in \mathbf{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$.
- If x > 0 and A is bounded above, then $\sup(xA) = x \sup A$.

proof:

- suppose $x \in \mathbf{R}$ and A is bounded above:
 - for all $a \in A$, we have $a \leq \sup A \implies x + a \leq x + \sup A$, *i.e.*, the set x + A is bounded by $x + \sup A$
 - let $\epsilon > 0$, for some $b \in A$, we have

$$\sup A - \epsilon < b \le \sup A \implies (x + \sup A) - \epsilon < x + b \le x + \sup A,$$

i.e.,
$$\sup(x+A) = x + \sup A$$

Real numbers

• suppose
$$x > 0$$
 and A is bounded above:

- for all $a \in A$, $a \leq \sup A \implies xa \leq x \sup A$, *i.e.*, the set xA is bounded by $x \sup A$ - let $\epsilon > 0 \implies \epsilon/x > 0$, for some $b \in A$, we have

 $\sup A - \epsilon/x < b \le \sup A \implies x \sup A - \epsilon < xb \le x \sup A,$

i.e., $\sup(xA) = x \sup A$

Remark 2.22 Similarly, we can also show that:

- If $x \in \mathbf{R}$ and A is bounded below, then $\inf(x + A) = x + \inf A$.
- If x > 0 and A is bounded below, then $\inf(xA) = x \inf A$.
- If x < 0 and A is bounded below, then $\sup(xA) = x \inf A$.
- If x < 0 and A is bounded above, then $\inf(xA) = x \sup A$.

Theorem 2.23 Let $A, B \subseteq \mathbf{R}$ where $x \leq y$ for all $x \in A$, $y \in B$, then $\sup A \leq \inf B$.

proof: for all $x \in A$, $y \in B$, $x \le y \implies B$ is bounded below by $x \implies x \le \inf B$ $\implies A$ is bounded above by $\inf B \implies \sup A \le \inf B$

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Absolute value

Definition 2.24 If $x \in \mathbf{R}$, we define the **absolute value** of x as

$$|x| = \begin{cases} x & x \ge 0\\ -x & x < 0. \end{cases}$$

Theorem 2.25

- $|x| \ge 0$, and, |x| = 0 if and only if x = 0.
- |-x| = |x| for all $x \in \mathbf{R}$.
- |xy| = |x||y| for all $x, y \in \mathbf{R}$.
- $|x|^2 = x^2$ for all $x \in \mathbf{R}$.
- $|x| \le y$ if and only if $-y \le x \le y$.
- $-|x| \le x \le |x|$ for all $x \in \mathbf{R}$.

Real numbers

Triangle inequality

Theorem 2.26 Triangle inequality. For all $x, y \in \mathbf{R}$,

 $|x+y| \le |x|+|y|.$

proof: let $x, y \in \mathbf{R}$

• $x + y \leq |x| + |y|$

•
$$-x + -y \le |-x| + |-y| = |x| + |y| \implies -(|x| + |y|) \le x + y$$

hence, we have

 $-(|x|+|y|) \leq x+y \leq |x|+|y| \implies |x+y| \leq |x|+|y|$

Corollary 2.27 Reverse triangle inequality. For all $x, y \in \mathbf{R}$,

$$\left||x| - |y|\right| \le |x - y|.$$

Real numbers

Definition 2.28 Let $x \in (0, 1]$ and let $d_{-i} \in \{0, 1, \dots, 9\}$. We say that x is represented by the digits $\{d_{-i} \mid i \in \mathbb{N}\}$, *i.e.*, $x = 0.d_{-1}d_{-2}\cdots$, if

$$x = \sup\{10^{-1}d_{-1} + 10^{-2}d_{-2} + \dots + 10^{-n}d_{-n} \mid n \in \mathbf{N}\}.$$

example: $0.2500 \dots = \sup\{\frac{2}{10}, \frac{2}{10} + \frac{5}{100}, \frac{2}{10} + \frac{5}{100} + \frac{0}{1000}, \dots\} = \sup\{\frac{1}{5}, \frac{1}{4}\} = \frac{1}{4}$

Theorem 2.29

- For all set of digits $\{d_{-i} \mid i \in \mathbb{N}\}$, there exists a unique $x \in [0, 1]$ such that $x = 0.d_{-1}d_{-2}\cdots$.
- For all $x \in (0,1]$, there exists a unique sequence of digits d_{-i} such that $x = 0.d_{-1}d_{-2}\cdots$ and

$$0.d_{-1}d_{-2}\cdots d_{-n} < x \le 0.d_{-1}d_{-2}\cdots d_{-n} + 10^{-n}$$
, for all $n \in \mathbb{N}$. (2.1)

• the second part indicates that the digital representation of 1/2 is $0.4999\cdots$

Real numbers

2-23

Theorem 2.30 Cantor. The set (0,1] is uncountable.

proof (by contradiction):

• assume (0,1] is countable, then there exists a bijection $x \colon \mathbf{N} \to (0,1]$, let

$$x(n) = 0.d_{-1}^{(n)}d_{-2}^{(n)}\cdots, \quad n \in \mathbf{N},$$

where $d_{-i}^{(n)}$ denotes the ith decimal of the real number $x(n) \in (0,1]$, and let

$$e_{-i} = \begin{cases} 1 & d_{-i}^{(i)} \neq 1 \\ 2 & d_{-i}^{(i)} = 1 \end{cases}$$
(2.2)

- let y = 0.e₋₁e₋₂ ··· , since all e_{-i} are nonzero, e₋₁, e₋₂,... satisfies (2.1); according to theorem 2.29, we have 0.e₋₁e₋₂ ··· being the unique decimal representation of y
- again according to theorem 2.29 and all e_{-i} are nonzero, we have $y \in (0,1] \implies \exists m \in \mathbf{N}, \ y = x(m) = 0.d_{-1}^{(m)}d_{-2}^{(m)} \cdots = 0.e_{-1}e_{-2}\cdots$, however, we have $e_{-m} \neq d_{-m}^{(m)}$ since (2.2), *i.e.*, for all $m \in \mathbf{N}, \ x(m) \neq y$, which is a contradiction

Corollary 2.31 The set of real numbers ${\bf R}$ is uncountable.

3. Sequences

- sequences and limits
- monotone sequences and subsequences
- inequalities and operations involving limits
- limit superior and limit inferior
- Bolzano-Weierstrass theorem
- Cauchy sequences

Sequences

Sequences and limits

Definition 3.1 A sequence (of real numbers) is a function $x: \mathbb{N} \to \mathbb{R}$. To denote a sequence we write $(x_n)_{n=1}^{\infty}$, where x_n is the *n*th element in the sequence.

• sequence need not start at n = 1, *e.g.*, the sequence $x \colon \{n \in \mathbb{Z} \mid n \ge m\} \to \mathbb{R}$ is denoted $(x_n)_{n=m}^{\infty}$

Definition 3.2 A sequence $(x_n)_{n=1}^{\infty}$ is **bounded** if there exists some $B \ge 0$ such that $|x_n| \le B$ for all $n \in \mathbf{N}$.

examples:

- the sequence $\left(\frac{1}{n}\right)_{n=1}^\infty$ is bounded since $\frac{1}{n} \leq 1$ for all n
- the sequence $(n)_{n=1}^{\infty}$ is not bounded since for all $B \ge 0$ there exists some $n \ge B$ according to the Archimedian property

Definition 3.3 A sequence $(x_n)_{n=1}^{\infty}$ is said to **converge** to $x \in \mathbf{R}$ if for all $\epsilon > 0$, there exists an $M \in \mathbf{N}$ such that for all $n \ge M$, we have $|x_n - x| < \epsilon$.

The number x is called a **limit** of the sequence. If the limit x is unique, we write

$$x = \lim_{n \to \infty} x_n.$$

A sequence that converges is said to be **convergent**, and otherwise is **divergent**.

Remark 3.4 A sequence $(x_n)_{n=1}^{\infty}$ is divergent if for all $x \in \mathbf{R}$, there exists some $\epsilon > 0$, such that for all $M \in \mathbf{N}$, there exists an $n \ge M$, so that $|x_n - x| \ge \epsilon$.

Theorem 3.5 Let $x, y \in \mathbf{R}$. If for all $\epsilon > 0$, $|x - y| < \epsilon$, then x = y.

proof: assume $x \neq y \implies |x - y| > 0$; take $\epsilon = \frac{1}{2}|x - y| \implies |x - y| < \frac{1}{2}|x - y| \implies |x - y| < \frac{1}{2}|x - y| \implies |x - y| < 0$, which is a contradiction

Theorem 3.6 If $(x_n)_{n=1}^{\infty}$ converges to x and y, then x = y, *i.e.*, a convergent sequence has a unique limit.

Sequences

proof: let $\epsilon > 0$

- $(x_n)_{n=1}^{\infty}$ converges to $x \implies \exists M_1 \in \mathbf{N}, \ \forall n \geq M_1, \ |x_n x| < \epsilon/2$
- $(x_n)_{n=1}^\infty$ converges to $y \implies \exists M_2 \in \mathbf{N}$, $\forall n \ge M_2$, $|x_n y| < \epsilon/2$
- let $M = M_1 + M_2$, then $M \ge M_1$ and $M \ge M_2$, then we have

$$|x_M - x| < \epsilon/2$$
 and $|x_M - y| < \epsilon/2$,

hence,

$$|x - y| = |(x - x_M) + (x_M - y)|$$

$$\leq |x - x_M| + |y - x_M|$$

$$< \epsilon/2 + \epsilon/2$$

$$= \epsilon$$

• according to theorem 3.5, we have x = y

Remark 3.7 Sometimes we write ' $x_n \to x$ as $n \to \infty$ ' to mean $x = \lim_{n \to \infty} x_n$. We may also avoid the 'as $n \to \infty$ ' part if the limiting process is clear from the context.

Example 3.8 Given the sequence $(x_n)_{n=1}^{\infty}$ with $x_n = c \in \mathbf{R}$ for all $n \in \mathbf{N}$, we have $\lim_{n\to\infty} x_n = c$.

proof: let $\epsilon > 0$, M = 1, then for all $n \ge M$, we have $|x_n - c| = |c - c| = 0 < \epsilon$

Example 3.9 The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to x = 0, *i.e.*, $\lim_{n \to \infty} \frac{1}{n} = 0$.

proof: let $\epsilon > 0$, choose an $M \in \mathbf{N}$ such that $M > 1/\epsilon$ (such an M exists according to the Archimedian property), then for all $n \ge M$, we have $\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| \le \frac{1}{M} < \epsilon$

Example 3.10 The sequence $\left(\frac{1}{n^2+2n+100}\right)_{n=1}^{\infty}$ converges to x = 0.

proof: let $\epsilon > 0$ choose $M \in \mathbf{N}$ such that $M \ge \epsilon^{-1}/2$, then for all $n \ge M$, we have

$$\left|\frac{1}{n^2 + 2n + 100} - 0\right| = \frac{1}{n^2 + 2n + 100} \le \frac{1}{2n} \le \frac{1}{2M} < \epsilon$$

Sequences

Example 3.11 The sequence $(x_n)_{n=1}^{\infty}$ where $x_n = (-1)^n$ is divergent.

proof: let $x \in \mathbf{R}$, $M \in \mathbf{N}$, then

$$|x_M - x_{M+1}| = \left| (-1)^M - (-1)^{M+1} \right| = 2$$

$$\implies 2 = |(x_M - x) + (x - x_{M+1})| \le |x_M - x| + |x_{M+1} - x|$$

$$\implies |x_M - x| \ge 1 \quad \text{or} \quad |x_{M+1} - x| \ge 1,$$

i.e., let $\epsilon=1,~n=M,$ we have either $|x_n-x|\geq\epsilon$ or $|x_{n+1}-x|\geq\epsilon$

Theorem 3.12 If $(x_n)_{n=1}^{\infty}$ is convergent, then $(x_n)_{n=1}^{\infty}$ is bounded.

proof:

- suppose $(x_n)_{n=1}^{\infty}$ converges to x, let $\epsilon = 1$, then there exists some $M \in \mathbb{N}$ such that for all $n \ge M$, $|x_n x| < 1 \implies x_n < |x| + 1$
- let $B = \max\{|x_1|, |x_2|, \dots, |x_M|, |x|+1\}$, since $x_n \leq |x_n|$ for all $n \in \mathbb{N}$, $n \leq M$, and $x_n < |x|+1$ for all $n \geq M$, we have $B \geq |x_n|$ for all $n \in \mathbb{N}$

Definition 3.13

- A sequence $(x_n)_{n=1}^{\infty}$ is monotone increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
- A sequence $(x_n)_{n=1}^{\infty}$ is monotone decreasing if $x_n \ge x_{n+1}$ for all $n \in \mathbb{N}$.
- If (x_n)[∞]_{n=1} is either monotone increasing or monotone decreasing, we say the sequence (x_n)[∞]_{n=1} is monotone (or monotonic).

examples:

- the sequence $\left(\frac{1}{n}\right)_{n=1}^\infty$ is monotone decreasing
- the sequence $\left(-\frac{1}{n}\right)_{n=1}^\infty$ is monotone increasing
- the sequence $((-1)^n)_{n=1}^\infty$ is not monotone

Sequences

Theorem 3.14 A monotone sequence $(x_n)_{n=1}^{\infty}$ converges if and only if it is bounded.

• If the sequence $(x_n)_{n=1}^{\infty}$ is monotone increasing and bounded, then

$$\lim_{n \to \infty} x_n = \sup\{x_n \mid n \in \mathbf{N}\}.$$

• If the sequence $(x_n)_{n=1}^{\infty}$ is monotone decreasing and bounded, then

$$\lim_{n \to \infty} x_n = \inf\{x_n \mid n \in \mathbf{N}\}.$$

proof: we prove for monotone increasing sequences, the other case is similar

- suppose $(x_n)_{n=1}^\infty$ is convergent, according to theorem 3.12, it is bounded
- suppose $(x_n)_{n=1}^\infty$ is monotone increasing and bounded
 - $(x_n)_{n=1}^{\infty}$ is monotone increasing $\implies x_n \leq x_{n+1}$ for all $n \in \mathbf{N}$
 - $-(x_n)_{n=1}^{\infty}$ is bounded \implies the set $\{x_n \mid n \in \mathbf{N}\}$ has supremum $x = \sup\{x_n \mid n \in \mathbf{N}\}$
 - let $\epsilon > 0$, according to theorem 2.17, there exists some $M \in \mathbf{N}$ such that $x \epsilon < x_M \leq x$, then for all $n \geq M$, we have

 $x - \epsilon < x_M \le x_n \le x < x + \epsilon \implies |x_n - x| < \epsilon$

Example

recall the following lemma from example 1.8 for the proof of the next theorem:

Lemma 3.15 Bernoulli's inequality. If $x \ge -1$ then $(x+1)^n \ge 1 + nx$ for all $n \in \mathbb{N}$.

Theorem 3.16 If $c \in (0,1)$ then the sequence $(c^n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} c^n = 0$. If c > 1, the sequence $(c^n)_{n=1}^{\infty}$ does not converge.

proof:

• if c > 1, we show that the sequence $(c^n)_{n=1}^{\infty}$ is unbounded (and hence does not converge):

- let $B \ge 0$, then there exists some $n \in \mathbf{N}$, $n > \frac{B}{c-1}$ such that

$$c^{n} = ((c-1)+1)^{n} \ge 1 + n(c-1) > n(c-1) > B$$

(the first inequality is because of lemma 3.15)

Sequences

• if $c \in (0,1)$, we first show that $(c^n)_{n=1}^{\infty}$ is monotone decreasing and bounded (and hence, convergent), *i.e.*, show that $c^{n+1} \leq c^n \leq c$ for all $n \in \mathbb{N}$ by induction:

– suppose $n=1\implies c^2\leq c\leq c,$ the first inequality holds since 0< c<1

– suppose n>1, and $c^{n+1}\leq c^n\leq c,$ then we have $c^{n+2}\leq c^{n+1}\leq c^n\leq c$

let $\lim_{n\to\infty} c^n = L$, we now show that L = 0

– let $\epsilon>0,$ then there exists some $M\in {\bf N}$ such that for all $n\geq M$ such that

$$|c^n - L| < \frac{1}{2}(1 - c)\epsilon$$

- hence, we have

$$\begin{aligned} (1-c)|L| &= |L-cL| \\ &= |(L-c^{M+1}) + (c^{M+1}-cL) \\ &\leq |L-c^{M+1}| + c|c^M - L| \\ &< |L-c^{M+1}| + |c^M - L| \\ &< \frac{1}{2}(1-c)\epsilon + \frac{1}{2}(1-c)\epsilon \\ &= (1-c)\epsilon, \end{aligned}$$

i.e., $|L| < \epsilon$ for all $\epsilon > 0$ (according to theorem 2.14) $\implies |L| \le 0 \implies L = 0$

Subsequences

Definition 3.17 Let $(x_n)_{n=1}^{\infty}$ be a sequence and $(n_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers. The sequence $(x_{n_i})_{i=1}^{\infty}$ is called a **subsequence** of $(x_n)_{n=1}^{\infty}$.

example: consider the sequence $(x_n)_{n=1}^{\infty} = (n)_{n=1}^{\infty}$, *i.e.*, 1, 2, 3, 4, ...

- the following are subsequences of $(x_n)_{n=1}^{\infty}$:
 - 1,3,5,7,9,11,..., described with $(x_{n_i})_{i=1}^{\infty} = (x_{2i-1})_{i=1}^{\infty}$
 - 2, 4, 6, 8, 10, 12, ..., described with $(x_{n_i})_{i=1}^{\infty} = (x_{2i})_{i=1}^{\infty}$
 - 2, 3, 5, 7, 11, 13, \ldots , described with $(x_{n_i})_{i=1}^\infty$ where n_i are primes
- the following are not subsequences of $(x_n)_{n=1}^{\infty}$:
 - $-1, 1, 1, 1, 1, 1, 1, \dots$
 - $-1, 1, 3, 3, 5, 5, \ldots$

Sequences

Theorem 3.18 If $\lim_{n\to\infty} x_n = x$, then all subsequences of $(x_n)_{n=1}^{\infty}$ converge to x.

proof:

- let $(x_{n_i})_{i=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$
- let $\epsilon > 0$, then there exists some $M_0 \in \mathbf{N}$ such that $|x_n x| < \epsilon$ for all $n \ge M_0$
- let $M = M_0$, then for all $i \ge M$, since $n_i \ge i \ge M = M_0$, we have

$$|x_{n_i} - x| < \epsilon$$

Remark 3.19 Theorem 3.18 implies that the sequence $((-1)^n)_{n=1}^{\infty}$ is divergent.

Theorem 3.20 The sequence $(x_n)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} x_n = x$ if and only if the sequence $(|x_n - x|)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} |x_n - x| = 0$.

proof: let $\epsilon > 0$

- suppose $\lim_{n\to\infty} x_n = x$, then $\exists M_0 \in \mathbb{N}$ such that $\forall n \ge M_0$, $|x_n x| < \epsilon$; let $M = M_0$, then $\forall n \ge M = M_0$, $|x_n x 0| = |x_n x| < \epsilon$
- suppose $\lim_{n\to\infty} |x_n x| = 0$, then $\exists M \in \mathbb{N}$, $\forall n \ge M$, $|x_n x 0| < \epsilon$, *i.e.*, $|x_n x| < \epsilon$

Theorem 3.21 Squeeze theorem. Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, and $(x_n)_{n=1}^{\infty}$ be sequences such that

 $a_n \le x_n \le b_n$

for all $n \in \mathbf{N}$. Suppose that $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ converge and

$$\lim_{n \to \infty} a_n = x = \lim_{n \to \infty} b_n.$$

Then $(x_n)_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} x_n = x$.

Sequences

proof: let $\epsilon > 0$

- $a_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \ge M_1$, $|a_n x| < \epsilon$
- $b_n \to x \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \ge M_2$, $|b_n x| < \epsilon$
- $a_n \le x_n \le b_n \implies a_n x \le x_n x \le b_n x$
- take $M = \max\{M_1, M_2\}$, then $\forall n \ge M$, we have

$$-\epsilon < a_n - x \le x_n - x \le b_n - x < \epsilon \implies |x_n - x| < \epsilon$$

Example 3.22 The sequence $\left(\frac{n^2}{n^2+n+1}\right)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1$.

proof:

• let $\epsilon > 0$, we have

$$0 \le \left| \frac{n^2}{n^2 + n + 1} - 1 \right| = \left| \frac{n + 1}{n^2 + n + 1} \right| \le \frac{n + 1}{n^2 + n} = \frac{1}{n}$$
$$0 \to 0 \text{ and } \frac{1}{n} \to 0 \implies \left| \frac{n^2}{n^2 + n + 1} - 1 \right| \to 0 \implies \frac{n^2}{n^2 + n + 1} \to 1$$

Theorem 3.23 Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences.

- If $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then we have $\lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n$.
- If $(x_n)_{n=1}^{\infty}$ converges and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq \lim_{n \to \infty} x_n \leq b$.

proof: we show the first statement since the second statement can then be proved by considering sequences $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ where $y_n = a \le x_n \le b = z_n$

- let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, suppose x > y
- $x > y \implies x y > 0$, let $\epsilon = \frac{x y}{2} > 0$
- $x_n \to x \implies \exists M_1 \in \mathbf{N} \text{ s.t. } \forall n \ge M_1, |x_n x| < \frac{x y}{2}$
- $y_n \to y \implies \exists M_2 \in \mathbf{N} \text{ s.t. } \forall n \geq M_2, \ |y_n y| < rac{x-y}{2}$
- let $M = \max\{M_1, M_2\}$, we have $x_M x > -\frac{x-y}{2}$ and $y_M y < \frac{x-y}{2}$, hence,

$$x_M > x - \frac{x - y}{2} = \frac{x + y}{2} = y + \frac{x - y}{2} > y_M,$$

which contradicts to $x_n \leq y_n$ for all $n \in \mathbf{N}$

Sequences

Operations involving limits

Theorem 3.24 Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

- The sequence $(x_n + y_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} (x_n + y_n) = x + y$.
- For all $c \in \mathbf{R}$, the sequence $(cx_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} cx_n = cx$.
- The sequence $(x_n y_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n y_n = xy$.
- If $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y \neq 0$, then the sequence $\left(\frac{x_n}{y_n}\right)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{x}{y}$.

proof:

• to show
$$x_n \to x, y_n \to y \implies x_n + y_n \to x + y$$
, let $\epsilon > 0$
 $-x_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \ge M_1, |x_n - x| < \epsilon/2$
 $-y_n \to y \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \ge M_2, |y_n - y| < \epsilon/2$
 $-$ let $M = \max\{M_1, M_2\}$, then for all $n \ge M$, we have
 $|(x_n + y_n) - (x + y)| \le |x_n - x| + |y_n - y| \le \epsilon/2 + \epsilon/2 = \epsilon$

 $\begin{array}{rcl} - & x_n \to x \implies \exists M \in \mathbf{N} \text{ such that } \forall n \geq M, \ |x_n - x| < \frac{1}{|c|+1}\epsilon \\ & - \text{ then for all } n \geq M, \text{ we have } |cx_n - cx| = |c||x_n - x| < \frac{|c|}{|c|+1}\epsilon < \epsilon \\ \bullet & \text{ we show that } x_n \to x, \ y_n \to y \implies x_ny_n \to xy: \end{array}$

• to show $x_n \to x \implies cx_n \to cx$, let $\epsilon > 0$

- We show that $x_n \to x$, $y_n \to y \implies x_n y_n \to xy$: - $x_n \to x \implies |x_n - x| \to 0$ - $y_n \to y \implies |y_n - y| \to 0$, and $(y_n)_{n=1}^{\infty}$ is bounded, *i.e.*, $\exists B \ge 0$, $|y_n| \le B$
 - hence, we have

$$0 \le |x_n y_n - xy| = |x_n y_n + xy_n - xy_n - xy|$$

= $|(x_n - x)y_n + (y_n - y)x|$
 $\le |x_n - x||y_n| + |y_n - y||x|$
 $\le |x_n - x|B + |y_n - y||x|$

- according to the previous statements, $|x_n x| \to 0 \implies |x_n x|B \to 0$, $|y_n y| \to 0 \implies |y_n y||x| \to 0$, then $|x_n x|B + |y_n y||x| \to 0$
- hence, according to theorem 3.21, $|x_ny_n xy|
 ightarrow 0$

Sequences

• to prove $x_n \to x$, $y_n \to y$ ($y_n \neq 0$ for all $n \in \mathbb{N}$, $y \neq 0$) $\implies \frac{x_n}{y_n} \to \frac{x}{y}$, we first show that there exists some b > 0 such that $|y_n| \ge b$:

$$- \text{ let } \epsilon = \tfrac{|y|}{2} \text{, then } y_n \to y \implies \exists M \in \mathbf{N} \text{ s.t. } \forall n \geq M \text{, } |y_n - y| < \tfrac{|y|}{2}$$

– then for all $n \ge M$, we have

$$\frac{|y|}{2} > |y_n - y| \ge ||y_n| - |y|| \implies |y_n| > \frac{|y|}{2}$$

(the second inequality is from the reverse triangle inequality)

- take $b = \min\{|y_1|, \dots, |y_M|, |y|/2\}$, we have $|y_n| \ge b$ for all $n \in \mathbb{N}$

we then show that $\left(\frac{1}{y_n}\right)_{n=1}^{\infty}$ converges with $\lim_{n\to\infty}\frac{1}{y_n}=\frac{1}{y}$: note that

$$0 \le \left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y_n - y}{y_n y}\right| = \frac{|y_n - y|}{|y_n||y|} \le \frac{|y_n - y|}{b|y|},$$

and $y_n \to y \implies \frac{|y_n - y|}{b|y|} \to 0$, hence, $\left|\frac{1}{y_n} - \frac{1}{y}\right| \to 0$, *i.e.*, $\frac{1}{y_n} \to \frac{1}{y}$ put together, $x_n \to x$ and $\frac{1}{y_n} \to \frac{1}{y} \implies \frac{x_n}{y_n} \to \frac{x}{y}$

Theorem 3.25 If $(x_n)_{n=1}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} x_n = x$, and $x_n \ge 0$ for all $n \in \mathbb{N}$, then the sequence $(\sqrt{x_n})_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

proof:

- suppose x = 0, let $\epsilon > 0$, then we have $x_n \to 0 \implies \exists M \in \mathbf{N} \text{ s.t. } \forall n \ge M$, $|x_n 0| = |x_n| < \epsilon^2 \implies \forall n \ge M$, $|\sqrt{x_n} \sqrt{x}| = |\sqrt{x_n}| < \sqrt{\epsilon^2} < \epsilon$
- suppose x > 0, we have

$$0 \le |\sqrt{x_n} - \sqrt{x}| = \left| \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} \right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \le \frac{|x_n - x|}{\sqrt{x}}$$

hence, $x_n \to x \implies |x_n - x| \to 0 \implies \frac{|x_n - x|}{\sqrt{x}} \to 0 \implies |\sqrt{x_n} - \sqrt{x}| \to 0$

Remark 3.26 Suppose the sequence $(x_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = x$. One can prove that $\lim_{n\to\infty} x_n^k = x^k$ by induction. Moreover, if $x_n \ge 0$ for all $n \in \mathbf{N}$, one can also prove that $\lim_{n\to\infty} \sqrt[k]{x_n} = \sqrt[k]{x}$.

Sequences

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Theorem 3.27 If $(x_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = x$, then $(|x_n|)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} |x_n| = |x|$.

proof: let $\epsilon > 0$

- $x_n \to x \implies \exists M \in \mathbf{N}$ such that $\forall n \ge M$, $|x_n x| < \epsilon$
- by reverse triangle inequality, for all $n \ge M$, we have

$$||x_n| - |x|| \le |x_n - x| < \epsilon$$

Theorem 3.28 If p > 0 then $\lim_{n\to\infty} n^{-p} = 0$.

proof: let $\epsilon > 0$, choose $M \in \mathbb{N}$ such that $M > (1/\epsilon)^{1/p}$, then for all $n \ge M$, $|n^{-p} - 0| = 1/n^p \le 1/M^p < \epsilon$

Theorem 3.29 If p > 0 then $\lim_{n \to \infty} p^{1/n} = 1$.

proof:

- if p = 1, $\lim_{n \to \infty} p^{1/n} = \lim_{n \to \infty} 1^{1/n} = 1$
- suppose p > 1- $p > 1 \implies p^{1/n} > 1^{1/n} = 1 \implies p^{1/n} - 1 > 0$
 - according to the Bernoulli's inequality (example 1.8), we have

$$\left(1 + (p^{1/n} - 1)\right)^n \ge 1 + n(p^{1/n} - 1) \implies \frac{p-1}{n} \ge p^{1/n} - 1 > 0$$

- $\frac{p-1}{n} \to 0 \implies p^{1/n} - 1 \to 0 \implies p^{1/n} \to 1$
if $0 1$, hence, $\lim_{n \to \infty} p^{1/n} = \lim_{n \to \infty} \frac{1}{(1/p)^{1/n}} = 1/1 = 1$

Sequences

Theorem 3.30 The sequence $(n^{1/n})_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} n^{1/n} = 1$.

proof:

- one can simply show that $n^{1/n} \geq 1$ by induction $\implies n^{1/n} 1 > 0$
- according to the binomial theorem, for all $x, y \in \mathbf{R}$ and $n \in \mathbf{N}$, we have $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- let x = 1, $y = n^{1/n} 1$, for all n > 1, we have

$$n = (1 + n^{1/n} - 1)^n = \sum_{k=0}^n \binom{n}{k} (n^{1/n} - 1)^k \ge \binom{n}{2} (n^{1/n} - 1)^2$$

$$\implies n \ge \frac{n!}{2!(n-2)!} (n^{1/n} - 1)^2 = \frac{1}{2} n(n-1)(n^{1/n} - 1)^2$$

$$\implies \sqrt{\frac{2}{n-1}} \ge n^{1/n} - 1 > 0$$

$$\implies n^{1/n} - 1 \to 0 \implies n^{1/n} \to 1$$

Definition 3.31 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Define, if the limits exist,

 $\limsup_{n \to \infty} x_n = \lim_{n \to \infty} (\sup\{x_k \mid k \ge n\}) \quad \text{and} \quad \liminf_{n \to \infty} x_n = \lim_{n \to \infty} (\inf\{x_k \mid k \ge n\}).$

They are called the limit superior and limit inferior, respectively.

Theorem 3.32 Let $(x_n)_{n=1}^\infty$ be a bounded sequence, and let

$$a_n = \sup\{x_k \mid k \ge n\}$$
 and $b_n = \inf\{x_k \mid k \ge n\}.$

Then:

- The sequence $(a_n)_{n=1}^{\infty}$ is monotone decreasing and bounded.
- The sequence $(b_n)_{n=1}^{\infty}$ is monotone increasing and bounded.
- We have $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Sequences

proof:

• we first prove the following lemma:

Lemma 3.33 Let $A, B \subseteq \mathbf{R}$, $A, B \neq \emptyset$, and A, B are bounded. If $A \subseteq B$ then we have $\inf B \leq \inf A \leq \sup A \leq \sup B$.

- $A \subseteq B \implies \sup B$ is an upper bound of $A \implies \sup A \leq \sup B$
- similarly, $\inf B$ is an lower bound of $A \implies \inf B \le \inf A$
- $-A, B \neq \emptyset \implies \inf A \leq \sup A \implies \inf B \leq \inf A \leq \sup A \leq \sup B$
- we now show the first two statements in the theorem
 - $(x_n)_{n=1}^\infty$ is bounded \implies there exists some $B\geq 0$ such that $-B\leq x_n\leq B$
 - for all $n \in \mathbb{N}$, we have $\{x_k \mid k \ge n+1\} \subseteq \{x_k \mid k \ge n\} \subseteq \{x_n \mid n \in \mathbb{N}\}$, according to lemma 3.33, this implies that

$$-B \le b_n \le b_{n+1} \le a_{n+1} \le a_n \le B,$$

i.e., $(a_n)_{n=1}^{\infty}$ is bounded monotone decreasing and $(b_n)_{n=1}^{\infty}$ is bounded monotone increasing ($\implies (a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge)

according to the previous inequalities, we have b_n ≤ a_n for all n ∈ N ⇒ lim_{n→∞} b_n ≤ lim_{n→∞} a_n (theorem 3.23), *i.e.*, lim inf_{n→∞} x_n ≤ lim sup_{n→∞} x_n

Example 3.34 We have $\limsup_{n\to\infty} (-1)^n = 1$ and $\liminf_{n\to\infty} (-1)^n = -1$.

proof: $\forall n \in \mathbf{N}$, the set $\{(-1)^k \mid k \ge n\} = \{-1, 1\} \implies \sup\{(-1)^k \mid k \ge n\} = 1$, $\inf\{(-1)^k \mid k \ge n\} = -1 \implies \limsup_{n \to \infty} (-1)^n = 1$ and $\liminf_{n \to \infty} (-1)^n = -1$

Example 3.35 We have $\limsup_{n\to\infty} \frac{1}{n} = \liminf_{n\to\infty} \frac{1}{n} = 0$.

proof: for all $n \in \mathbb{N}$, we have $\sup\{1/k \mid k \ge n\} = 1/k$ and $\inf\{1/k \mid k \ge n\} = 0$, hence,

$$\limsup_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \frac{1}{k} = 0 \quad \text{and} \quad \liminf_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} 0 = 0$$

Sequences

Bolzano-Weierstrass theorem

Theorem 3.36 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Then, there exists subsequences $(x_{n_i})_{i=1}^{\infty}$ and $(x_{m_i})_{i=1}^{\infty}$ such that

$$\lim_{i \to \infty} x_{n_i} = \limsup_{n \to \infty} x_n \quad \text{and} \quad \lim_{i \to \infty} x_{m_i} = \liminf_{n \to \infty} x_n.$$

proof: let $a_n = \sup\{x_k \mid k \ge n\}$

- $a_1 = \sup\{x_k \mid k \ge 1\} \implies \exists n_1 \ge 1 \text{ such that } a_1 1 < x_{n_1} \le a_1$
- $a_{n_1+1} = \sup\{x_k \mid k \ge n_1+1\} \implies \exists n_2 > n_1 \text{ s.t. } a_{n_1+1} \frac{1}{2} < x_{n_2} \le a_{n_1+1}$
- $a_{n_2+1} = \sup\{x_k \mid k \ge n_2+1\} \implies \exists n_3 > n_1 \text{ s.t. } a_{n_2+1} \frac{1}{3} < x_{n_3} \le a_{n_2+1}$
- repeatedly, we can find a sequence of integers $n_1 < n_2 < \cdots$ such that

$$a_{n_{i-1}+1} - \frac{1}{i} < x_{n_i} \le a_{n_{i-1}+1}$$

(defining $n_0 = 0$)

- $(a_{n_{i-1}+1})_{i=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$, and $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$ $\implies \lim_{n\to\infty} a_{n_{i-1}+1} = \limsup_{n\to\infty} x_n \implies \lim_{n\to\infty} x_{n_i} = \limsup_{n\to\infty} x_n$
- similarly, we can find a subsequence of $(x_n)_{n=1}^{\infty}$ that converges to $\liminf_{n\to\infty} x_n$

Sequences

Theorem 3.37 *Bolzano-Weierstrass.* Every bounded sequence consisting of real numbers has a convergent subsequence.

Theorem 3.38 Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence. Then, $(x_n)_{n=1}^{\infty}$ converges if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

proof:

- suppose $\lim_{n\to\infty} x_n = x$, then the subsequences that converge to $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ must converge to x (theorem 3.18)
- suppose $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$, for all $n \in \mathbb{N}$, according to the squeeze theorem,

$$\inf\{x_k \mid k \ge n\} \le x_n \le \sup\{x_k \mid k \ge n\} \implies \lim_{n \to \infty} x_n = x$$

Sequences

Cauchy sequences

Definition 3.39 A sequence $(x_n)_{n=1}^{\infty}$ is **Cauchy** if for all $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $n, k \ge M$, we have $|x_n - x_k| < \epsilon$.

Remark 3.40 A sequence $(x_n)_{n=1}^{\infty}$ is not Cauchy if there exists some $\epsilon > 0$, such that for all $M \in \mathbf{N}$, there exists some $n, k \ge M$, so that $|x_n - x_k| \ge \epsilon$.

Example 3.41 The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is Cauchy.

proof: let $\epsilon > 0$, choose $M \in \mathbf{N}$ such that $M > 2/\epsilon$, then for all $n, k \ge M$, we have

$$\left|\frac{1}{n} - \frac{1}{k}\right| \le \frac{1}{n} + \frac{1}{k} \le \frac{2}{M} < \epsilon$$

Example 3.42 The sequence $((-1)^n)_{n=1}^{\infty}$ is not Cauchy.

proof: let
$$\epsilon = 1$$
, $M \in \mathbf{N}$, $n = M$, $k = M + 1$, then $\left| \left(-1 \right)^n - \left(-1 \right)^k \right| = 2 \ge \epsilon$

Sequences
Theorem 3.43 If the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy, then $(x_n)_{n=1}^{\infty}$ is bounded.

proof:

- let $\epsilon = 1$, $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies \exists M \in \mathbf{N}$ such that $\forall n, k \geq M$, $|x_n x_k| < 1$
- let $k = M \implies \forall n \ge M$, $|x_n x_M| < 1 \implies \forall n \ge M$, $|x_n| < |x_M| + 1$
- take $B = \max\{|x_1|, |x_2|, ..., |x_M|, |x_M| + 1\}$, then $|x_n| \le B$ for all $n \in \mathbb{N}$

Theorem 3.44 If the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy and a subsequence $(x_{n_i})_{i=1}^{\infty}$ converges, then $(x_n)_{n=1}^{\infty}$ converges.

proof: let $\epsilon > 0$

- $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies \exists M_1 \in \mathbf{N}$ such that $\forall n, k \geq M_1$, $|x_n x_k| < \epsilon/2$
- let $\lim_{i\to\infty} x_{n_i} = x \implies \exists M_2 \in \mathbf{N}$ such that $\forall i \ge M_2$, $|x_{n_i} x| < \epsilon/2$
- let $M = \max\{M_1, M_2\}$, then $\forall k \ge M$, $n_k \ge k \ge M_1$, $n_k \ge k \ge M_2$, hence,

$$|x_k - x| \le |x_k - x_{n_k}| + |x_{n_k} - x| < \epsilon/2 + \epsilon/2 = \epsilon$$

Sequences

Theorem 3.45 Completeness of the real numbers. A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is Cauchy if and only if the sequence $(x_n)_{n=1}^{\infty}$ is convergent.

proof:

- suppose $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies (x_n)_{n=1}^{\infty}$ is bounded (theorem 3.43) \implies there exists convergent subsequence of $(x_n)_{n=1}^{\infty}$ (theorem 3.37) $\implies (x_n)_{n=1}^{\infty}$ is convergent (theorem 3.44)
- suppose $\lim_{n\to\infty} x_n = x$, let $\epsilon > 0$, then $\exists M \in \mathbf{N}$, $\forall n \ge M$, $|x_n x| < \epsilon/2$; let $k \ge M$, then $|x_n x_k| \le |x_n x| + |x x_k| < \epsilon/2 + \epsilon/2 = \epsilon$

Remark 3.46 We say a set is **Cauchy-complete**, or just **complete**, if all Cauchy sequence of elements in the set converges to some point in the set. Theorem 3.45 indicates that \mathbf{R} is complete.

Remark 3.47 The set \mathbf{Q} is *not* complete. Since \mathbf{Q} does not have the least upper bound property, then, *e.g.*, $\sup\{x_n \mid n \in \mathbf{N}\}$, $\sup\{x_k \mid k \ge n\}$, *etc.*, might not exist in \mathbf{Q} .

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4. Series

- series
- Cauchy series
- linearity of series
- absolute convergence
- comparison, ratio, and root tests
- alternating series
- rearrangements

Series

Series

Definition 4.1 Given a sequence $(x_n)_{n=1}^{\infty}$, the formal object $\sum_{n=1}^{\infty} x_n$ is called a series. A series **converges** if the sequence $(s_m)_{m=1}^{\infty}$ defined by

$$s_m = \sum_{n=1}^m x_n = x_1 + \dots + x_m$$

converges. The numbers s_m are called **partial sums**. If the series converges, we write

$$\sum_{n=1}^{\infty} x_n = \lim_{m \to \infty} s_m.$$

In this case, we treat $\sum_{n=1}^{\infty} x_n$ as a number.

If the sequence $(s_m)_{m=1}^{\infty}$ diverges, we say the series is **divergent**. In this case, $\sum_{n=1}^{\infty} x_n$ is simply a formal object and not a number.

• series need not start at n = 1

Example 4.2 The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

proof: the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is given by:

$$s_m = \sum_{n=1}^m \frac{1}{n(n+1)}$$

= $\sum_{n=1}^m \frac{1}{n} - \frac{1}{n+1}$
= $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{m} - \frac{1}{m+1}$
= $1 - \frac{1}{m+1}$,

hence, $s_m \to 1 \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Series

4-3

Theorem 4.3 If |r| < 1, then $\sum_{n=0}^{\infty} r^n$ converges and $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

proof:

• the sequence of partial sums $(s_m)_{m=1}^\infty$ is given by:

$$s_m = \sum_{n=0}^m r^n = \frac{\left(\sum_{n=0}^m r^n\right)(1-r)}{1-r} = \frac{\sum_{n=0}^m (r^n - r^{n+1})}{1-r} = \frac{1-r^{m+1}}{1-r}$$

• $|r| < 1 \implies r^n \to 0$ (theorem 3.16) $\implies s_m \to \frac{1}{1-r}$

Remark 4.4 Series of the form $\sum_{n=0}^{\infty} \alpha r^n$ with $\alpha, r \in \mathbf{R}$ are called **geometric series**.

Theorem 4.5 Let $(x_n)_{n=1}^{\infty}$ be a sequence and let $M \in \mathbb{N}$. Then, $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=M}^{\infty} x_n$ converges.

proof:

• for all $m \ge M$, we have

$$\sum_{n=1}^{m} x_n = \sum_{n=1}^{M-1} x_n + \sum_{n=M}^{m} x_n$$

• suppose $\sum_{n=1}^{\infty} x_n$ converges, we have

$$\lim_{m \to \infty} \sum_{n=M}^{m} x_n = \lim_{m \to \infty} \left(\sum_{n=1}^{m} x_n - \sum_{n=1}^{M-1} x_n \right) = \lim_{m \to \infty} \left(\sum_{n=1}^{m} x_n \right) - \sum_{n=1}^{M-1} x_n$$

• suppose $\sum_{n=M}^{\infty} x_n$ converges, we have

$$\lim_{m \to \infty} \sum_{n=1}^m x_n = \lim_{m \to \infty} \left(\sum_{n=M}^m x_n + \sum_{n=1}^{M-1} x_n \right) = \lim_{m \to \infty} \left(\sum_{n=M}^m x_n \right) + \sum_{n=1}^{M-1} x_n$$

Series

Cauchy series

Definition 4.6 The series $\sum_{n=1}^{\infty} x_n$ is **Cauchy** if the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is Cauchy.

Theorem 4.7 The series $\sum_{n=1}^{\infty} x_n$ is Cauchy if and only if $\sum_{n=1}^{\infty} x_n$ is convergent.

proof: according to theorem 3.45

- suppose $\sum_{n=1}^{\infty} x_n$ is Cauchy $\implies (s_m)_{m=1}^{\infty}$ is Cauchy $\implies (s_m)_{m=1}^{\infty}$ is convergent $\implies \sum_{n=1}^{\infty} x_n$ is convergent
- suppose $\sum_{n=1}^{\infty} x_n$ is convergent $\implies (s_m)_{m=1}^{\infty}$ is convergent $\implies (s_m)_{m=1}^{\infty}$ is Cauchy $\implies \sum_{n=1}^{\infty} x_n$ is Cauchy

Theorem 4.8 The series $\sum_{n=1}^{\infty} x_n$ is Cauchy if and only if for all $\epsilon > 0$, there exists an $M \in \mathbf{N}$ such that for all $m \ge M$ and k > m, we have $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$.

proof: let $\epsilon > 0$

• suppose $\sum_{n=1}^{\infty} x_n$ is Cauchy $\implies (\sum_{n=1}^m x_n)_{m=1}^{\infty}$ is Cauchy $\implies \exists M \in \mathbb{N}$ such that $\forall m, k \ge M$ (assume k > m), we have

$$\left|\sum_{n=1}^{m} x_n - \sum_{n=1}^{k} x_n\right| < \epsilon \implies \left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$$

• suppose $\exists M \in \mathbf{N}$ such that for all $k > m \ge M$, $\left| \sum_{n=m+1}^{k} x_n \right| < \epsilon$, then we have

$$\left|\sum_{n=1}^{m} x_n - \sum_{n=1}^{k} x_n\right| = \left|\sum_{n=m+1}^{k} x_n\right| < \epsilon,$$

i.e., $(\sum_{n=1}^m x_n)_{m=1}^\infty$ is Cauchy $\implies \sum_{n=1}^\infty x_n$ is Cauchy

Series

Theorem 4.9 If the series $\sum_{n=1}^{\infty} x_n$ converges then $\lim_{n\to\infty} x_n = 0$.

proof: let $\epsilon > 0$, $\sum_{n=1}^{\infty} x_n$ converges $\implies \sum_{n=1}^{\infty} x_n$ is Cauchy $\implies \exists M_0 \in \mathbf{N}$ such that $\forall k > m \ge M_0$, we have $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$ (theorem 4.8); choose $M = M_0 + 1$, then $\forall m \ge M$, by taking $k = m > m - 1 \ge M_0$, we have

$$|x_m - 0| = |x_m| = \left|\sum_{n=m-1+1}^m x_n\right| < \epsilon \implies \lim_{n \to \infty} x_n = 0$$

Remark 4.10 The converse of theorem 4.9 does not hold.

Theorem 4.11 If $|r| \ge 1$ then the series $\sum_{n=0}^{\infty} r^n$ diverges.

proof: If $|r| \ge 1$, then $\lim_{n\to\infty} r^n \ne 0$, according to theorem 4.9, $\sum_{n=0}^{\infty} r^n$ diverges

Corollary 4.12 The series $\sum_{n=0}^{\infty} \alpha r^n$ with $\alpha, r \in \mathbf{R}$ converges if and only if |r| < 1.

Theorem 4.13 The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

proof: we show that a subsequence of $(s_m)_{m=1}^{\infty}$ is unbounded

• consider the subsequence $(s_{2^i})_{i=1}^{\infty}$, given by

$$s_{2^{i}} = \sum_{n=1}^{2^{i}} \frac{1}{n} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{i-1} + 1} + \dots + \frac{1}{2^{i}}\right)$$
$$= 1 + \sum_{k=1}^{i} \sum_{n=2^{k-1}+1}^{2^{k}} \frac{1}{n}$$
$$\ge 1 + \sum_{k=1}^{i} \sum_{n=2^{k-1}+1}^{2^{k}} \frac{1}{2^{k}} = 1 + \sum_{k=1}^{i} \frac{1}{2^{k}} (2^{k} - (2^{k-1} + 1) + 1)$$
$$= 1 + \sum_{k=1}^{i} \frac{2^{k-1}}{2^{k}} = 1 + \sum_{k=1}^{i} \frac{1}{2} = 1 + \frac{i}{2}$$

• $(1+i/2)_{i=1}^{\infty}$ is unbounded $\implies (s_{2^i})_{i=1}^{\infty}$ is unbounded $\implies (s_m)_{m=1}^{\infty}$ is unbounded $\implies \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge

Series

Linearity of series

Theorem 4.14 Let $\alpha \in \mathbf{R}$ and $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series. Then the series $\sum_{n=1}^{\infty} (\alpha x_n + y_n)$ converges and

$$\sum_{n=1}^{\infty} (\alpha x_n + y_n) = \alpha \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$$

proof: consider the partial sums of $\sum_{n=1}^{\infty} (\alpha x_n + y_n)$, we have

$$\sum_{n=1}^{m} (\alpha x_n + y_n) = \alpha \sum_{n=1}^{m} x_n + \sum_{n=1}^{m} y_n$$

$$\implies \lim_{m \to \infty} \sum_{n=1}^{m} (\alpha x_n + y_n) = \alpha \lim_{m \to \infty} \sum_{n=1}^{m} x_n + \lim_{m \to \infty} \sum_{n=1}^{m} y_n$$

$$\implies \sum_{n=1}^{\infty} (\alpha x_n + y_n) = \alpha \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$$

Theorem 4.15 If $x_n \ge 0$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is bounded.

proof:

- suppose $\sum_{n=1}^{\infty} x_n$ converges $\implies (s_m)_{m=1}^{\infty}$ converges $\implies (s_m)_{m=1}^{\infty}$ is bounded
- suppose $(s_m)_{m=1}^{\infty}$ is bounded, since $x_n \ge 0$ for all $n \in \mathbf{N}$, we have

$$s_m = \sum_{n=1}^m x_n \le \sum_{n=1}^m x_n + x_{n+1} = s_{m+1},$$

i.e., $(s_m)_{m=1}^{\infty}$ is monotone increasing $\implies (s_m)_{m=1}^{\infty}$ converges $\implies \sum_{n=1}^{\infty} x_n$ converges

Definition 4.16 The series $\sum_{n=1}^{\infty} x_n$ converges absolutely if $\sum_{n=1}^{\infty} |x_n|$ converges.

Series

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Theorem 4.17 If the series $\sum_{n=1}^{\infty} x_n$ converges absolutely then $\sum_{n=1}^{\infty} x_n$ converges.

proof:

• we first prove the following claim by induction:

Lemma 4.18 For all $x_1, \ldots, x_n \in \mathbf{R}$, we have $|\sum_{i=1}^n x_i| \le \sum_{i=1}^n |x_i|$.

- suppose n = 2, we have the triangle inequality $|x_1 + x_2| \le |x_1| + |x_2|$
- suppose n>2, and $|\sum_{i=1}^n x_i| \leq \sum_{i=1}^n |x_i|$ holds, we have

$$\left|\sum_{i=1}^{n+1} x_i\right| \le \left|\sum_{i=1}^n x_i\right| + |x_{n+1}| \le \sum_{i=1}^n |x_i| + |x_{n+1}| = \sum_{i=1}^{n+1} |x_i|$$

- $\sum_{n=1}^{\infty} x_n$ converges absolutely $\implies \sum_{n=1}^{\infty} |x_n|$ converges \implies let $\epsilon > 0$, $\exists M \in \mathbf{N} \text{ s.t. } \forall k > m \ge M$, $|\sum_{n=m+1}^{k} |x_n|| = \sum_{n=m+1}^{k} |x_n| < \epsilon$
- hence, for all $k > m \ge M$, we have $\left|\sum_{n=m+1}^{k} x_n\right| \le \sum_{n=m+1}^{k} |x_n| < \epsilon \implies \sum_{n=1}^{\infty} x_n$ converges

Remark 4.19 The converse of theorem 4.17 does not hold.

Theorem 4.20 Comparison test. Suppose $0 \le x_n \le y_n$ for all $n \in \mathbb{N}$.

- If $\sum_{n=1}^{\infty} y_n$ converges then $\sum_{n=1}^{\infty} x_n$ converges.
- If $\sum_{n=1}^{\infty} x_n$ diverges then $\sum_{n=1}^{\infty} y_n$ diverges.

proof:

• suppose
$$\sum_{n=1}^{\infty} y_n$$
 converges $\implies (\sum_{n=1}^m y_n)_{m=1}^{\infty}$ is bounded $\implies \exists B \ge 0$ s.t.
 $\forall m \in \mathbf{N}, |\sum_{n=1}^m y_n| = \sum_{n=1}^m y_n \le B \implies \forall m \in \mathbf{N}, \text{ we have}$
 $0 \le \sum_{n=1}^m x_n \le \sum_{n=1}^m y_n \le B$
 $\implies (\sum_{n=1}^m x_n)_{m=1}^{\infty}$ is bounded $\implies \sum_{n=1}^{\infty} x_n$ converges (theorem 4.15)
• suppose $\sum_{n=1}^{\infty} x_n$ diverges $\implies (\sum_{n=1}^m x_n)_{m=1}^{\infty}$ is unbounded (theorem 4.15)
 $\implies \forall B \ge 0, \exists m \in \mathbf{N} \text{ such that } |\sum_{n=1}^m x_n| = \sum_{n=1}^m x_n > B, \text{ hence, for this } m,$
 $\sum_{n=1}^m y_n \ge \sum_{n=1}^m x_n > B$
 $\implies (\sum_{n=1}^m y_n)_{m=1}^{\infty}$ is unbounded $\implies \sum_{n=1}^{\infty} y_n$ diverges
 $\Rightarrow (\sum_{n=1}^m y_n)_{m=1}^{\infty}$ is unbounded $\implies \sum_{n=1}^{\infty} y_n$ diverges

Series

Theorem 4.21 For $p \in \mathbf{R}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if p > 1.

proof:

- suppose $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, assume $p \leq 1$, then we have $0 < \frac{1}{n} \leq \frac{1}{n^p}$; the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\implies \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (theorem 4.20), which is a contradiction
- suppose p > 1, let $s_m = \sum_{n=1}^m \frac{1}{n^p}$ we first show that $s_m \le s_{2^m}$ for all $m \in \mathbb{N}$: by induction, we have $2^m > m$ for all $m \in \mathbb{N} \implies s_m = \sum_{n=1}^m \frac{1}{n^p} \le \sum_{n=1}^{2^m} \frac{1}{n^p} = s_{2^m}$
 - we now show that s_{2^m} is bounded by $1 + \frac{1}{1-2^{-(p-1)}}$:

$$s_{2^m} = \sum_{n=1}^{2^m} \frac{1}{n^p}$$

= $1 + \left(\frac{1}{2^p}\right) + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \dots + \left(\frac{1}{(2^{m-1}+1)^p} + \dots + \frac{1}{(2^m)^p}\right)$
= $1 + \sum_{k=1}^m \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n^p} \le 1 + \sum_{k=1}^m \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{(2^{k-1}+1)^p}$

$$\leq 1 + \sum_{k=1}^{m} \sum_{n=2^{k-1}+1}^{2^{n}} \frac{1}{(2^{k-1})^{p}} = 1 + \sum_{k=1}^{m} 2^{-p(k-1)} (2^{k} - (2^{k-1}+1)+1)$$
$$= 1 + \sum_{k=1}^{m} 2^{-(p-1)(k-1)} = 1 + \sum_{k=0}^{m-1} 2^{-(p-1)k}$$
$$\leq 1 + \sum_{k=0}^{\infty} 2^{-(p-1)k} = 1 + \sum_{k=0}^{\infty} \left(2^{-(p-1)}\right)^{k}$$
$$= 1 + \frac{1}{1 - 2^{-(p-1)}},$$

where the last equality is from the fact that p-1 > 0, and using the properties of geometric series (theorem 4.3)

- put together, we have $0 < s_m \le s_{2^m} \le 1 + \frac{1}{1-2^{-(p-1)}} \implies (s_m)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies (s_m)_{m=1}^{\infty}$ converges $\implies \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

Series

Ratio test

Theorem 4.22 Ratio test. Suppose $x_n \neq 0$ for all n and the limit

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$$L = \lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|}$$

exists.

- If L > 1 then $\sum_{n=1}^{\infty} x_n$ diverges.
- If L < 1 then $\sum_{n=1}^{\infty} x_n$ converges absolutely.

proof:

• suppose L > 1, then $\exists M \in \mathbf{N}$ such that $\forall n \ge M$, $\frac{|x_{n+1}|}{|x_n|} \ge 1 \implies \forall n \ge M$, $|x_{n+1}| \ge |x_n| \implies \lim_{n \to \infty} x_n \ne 0 \implies \sum_{n=1}^{\infty} x_n$ diverges (theorem 4.9)

• suppose
$$L < 1$$
, let $L < \alpha < 1$
 $- \exists M \in \mathbf{N}$ such that $\forall n \ge M$, $\frac{|x_{n+1}|}{|x_n|} \le \alpha \implies \forall n \ge M$, $|x_{n+1}| \le \alpha |x_n| \implies$

$$|x_n| \le \alpha |x_{n-1}| \le \alpha^2 |x_{n-2}| \le \dots \le \alpha^{n-M} |x_M| \implies |x_n| \le \alpha^{n-M} |x_M|, \ \forall n \ge M$$

– consider the partial sums of the series $\sum_{n=1}^\infty |x_n|$, assume m>M , we have

$$\sum_{n=1}^{m} |x_n| = \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{m} |x_n| \le \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} |x_n|$$
$$\le \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} \alpha^{n-M} |x_M| = \sum_{n=1}^{M-1} |x_n| + |x_M| \sum_{n=0}^{\infty} \alpha^n$$
$$= \sum_{n=1}^{M-1} |x_n| + \frac{|x_M|}{1 - \alpha},$$

where the last equality is from the properties of geometric series and $0 < \alpha < 1$

- hence, the sequence of partial sums $(\sum_{n=1}^{m} |x_n|)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies \sum_{n=1}^{\infty} |x_n|$ converges $\implies \sum_{n=1}^{\infty} x_n$ converges absolutely

Remark 4.23 If L = 1 in theorem 4.22 then the test doesn't apply. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Series

4-17

Example 4.24 The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ converges absolutely.

proof:

$$\left|\frac{(-1)^n}{n^2+1}\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \implies \lim_{n \to \infty} \frac{\left|\frac{(-1)^{n+1}}{(n+1)^2+1}\right|}{\left|\frac{(-1)^n}{n^2+1}\right|} < \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 1$$

Example 4.25 The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbf{R}$.

proof:

$$\lim_{n \to \infty} \frac{\left| \frac{x^{n+1}}{(n+1)!} \right|}{\left| \frac{x^n}{n!} \right|} = \lim_{n \to \infty} \frac{|x|}{n+1} = 0 < 1$$

Theorem 4.26 Root test. Let $\sum_{n=1}^{\infty} x_n$ be a series and suppose that the limit

$$L = \lim_{n \to \infty} |x_n|^{1/n}$$

exists.

- If L > 1 then $\sum_{n=1}^{\infty} x_n$ diverges.
- If L < 1 then $\sum_{n=1}^{\infty} x_n$ converges absolutely.

proof:

- suppose L > 1, then $\exists M \in \mathbf{N}$ s.t. $\forall n \ge M$, $|x_n|^{1/n} \ge 1 \implies \forall n \ge M$, $|x_n| \ge 1$ $\implies \lim_{n \to \infty} x_n \ne 0 \implies \sum_{n=1}^{\infty} x_n$ diverges (theorem 4.9)
- suppose L < 1, let $L < \alpha < 1$ $- \exists M \in \mathbf{N}$ such that $\forall n \ge M$, $|x_n|^{1/n} \le \alpha \implies \forall n \ge M$, $|x_n| \le \alpha^n$

Series

_	consider [•]	the partial	sums of	the series	$\sum_{n=1}^{\infty} x_n ,$	assume $m >$	$\cdot M$, we have	/e
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$$\sum_{n=1}^{m} |x_n| = \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{m} |x_n| \le \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} |x_n|$$
$$\le \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} \alpha^n = \sum_{n=1}^{M-1} |x_n| + \sum_{n=0}^{\infty} \alpha^{M+n}$$
$$= \sum_{n=1}^{M-1} |x_n| + \alpha^M \sum_{n=0}^{\infty} \alpha^n$$
$$= \sum_{n=1}^{M-1} |x_n| + \frac{\alpha^M}{1-\alpha},$$

where the last equality is from the properties of geometric series and $0<\alpha<1$

- hence, the sequence of partial sums $(\sum_{n=1}^{m} |x_n|)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies \sum_{n=1}^{\infty} |x_n|$ converges $\implies \sum_{n=1}^{\infty} x_n$ converges absolutely

Remark 4.27 Similarly, if L = 1 in theorem 4.26 then the test doesn't apply.

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Alternating series

Theorem 4.28 Let $(x_n)_{n=1}^{\infty}$ be a monotone decreasing sequence with $\lim_{n\to\infty} x_n = 0$. Then the series $\sum_{n=1}^{\infty} (-1)^n x_n$ converges.

proof: consider the partial sums of $\sum_{n=1}^{\infty} (-1)^n x_n$, given by $s_m = \sum_{n=1}^m (-1)^n x_n$

- $(x_n)_{n=1}^{\infty}$ is monotone decreasing and $x_n \to 0 \implies \forall n \in \mathbb{N}, x_n \ge x_{n+1} \ge 0$
- we first show that the subsequence $(s_{2m})_{m=1}^{\infty}$ converges, notice that

$$s_{2m} = \sum_{n=1}^{2m} (-1)^n x_n = -x_1 + x_2 - x_3 + \dots - x_{2m-1} + x_{2m}$$
(4.1)

- rearranging the terms in (4.1), since $x_{n+1} \leq x_n$, $\forall n \in \mathbf{N}$, we have

$$s_{2m} = (x_2 - x_1) + (x_4 - x_3) + \dots + (x_{2m} - x_{2m-1})$$

$$\geq (x_2 - x_1) + (x_3 - x_2) + \dots + (x_{2m} - x_{2m-1}) + (x_{2m+2} - x_{2m+1})$$

$$= s_{2(m+1)}$$

 $\implies (s_{2m})_{m=1}^{\infty}$ is monotone decreasing

Series

– r	earranging the	terms in (4.1)	differently,	since x_n	$\geq x_{n+1}$	≥ 0 ,	, $orall n \in \mathbf{N}$,	we ha	ive
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$$s_{2m} = -x_1 + (x_2 - x_3) + (x_4 - x_5) + \dots + (x_{2m-2} - x_{2m-1}) + x_{2m} \ge -x_1$$

 $\implies (s_{2m})_{m=1}^{\infty}$ is bounded below

- put together, we conclude that $(s_{2m})_{m=1}^\infty$ converges, let $s_{2m} o x$
- we now show that $(s_m)_{m=1}^{\infty}$ also converges to x, let $\epsilon > 0$
 - $s_{2m} o x \implies \exists M_1 \in \mathbf{N}$ such that $orall m \geq M_1$, $|s_{2m} x| < \epsilon/2$

 $-x_n o 0 \implies \exists M_2 \in \mathbf{N} \text{ such that } \forall m \geq M_2, \ |x_m| < \epsilon/2$

let $M = \max\{2M_1 + 1, M_2\}$, then $\forall m \ge M$, $m \ge 2M_1 + 1$ and $m \ge M_2$ - if m is even $\implies \frac{m}{2} > M_1$, hence

$$|s_m - x| = \left|s_{2 \cdot \frac{m}{2}} - x\right| < \epsilon/2 < \epsilon$$

- if m is odd, then m-1 is even and $m-1\geq 2M_1\implies \frac{m-1}{2}\geq M_1$, hence

$$|s_m - x| = |s_{m-1} - x + x_m| = \left|s_{2 \cdot \frac{m-1}{2}} - x + x_m\right|$$
$$\leq \left|s_{2 \cdot \frac{m-1}{2}} - x\right| + |x_m| < \epsilon/2 + \epsilon/2 = \epsilon$$

put together, we have $(s_m)_{m=1}^{\infty}$ converges $\implies \sum_{n=1}^{\infty} (-1)^n x_n$ converges

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Corollary 4.29 The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but does not converge absolutely.

proof:

- since $(\frac{1}{n})_{n=1}^{\infty}$ is monotone decreasing with $\lim_{n\to\infty} \frac{1}{n} = 0$, it follows immediately from theorem 4.28 that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges
- since $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not converge absolutely

Series

4-23

Rearrangements

Theorem 4.30 Suppose $\sum_{n=1}^{\infty} x_n$ converges absolutely and $\sum_{n=1}^{\infty} x_n = x$. Let $\sigma \colon \mathbf{N} \to \mathbf{N}$ be a bijective function. Then, the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is absolutely convergent and $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$. In other words, absolute convergence implies, if we rearrange the sequence, the new series will still converge to the same value of the original series.

proof:

• we first show $\sum_{n=1}^{\infty} |x_{\sigma(n)}|$ converges, *i.e.*, $\left(\sum_{n=1}^{m} |x_{\sigma(n)}|\right)_{m=1}^{\infty}$ is bounded - $\sum_{n=1}^{\infty} |x_n|$ converges $\implies \left(\sum_{n=1}^{m} |x_n|\right)_{m=1}^{\infty}$ is bounded $\implies \exists B \ge 0$ such that $\forall m \in \mathbf{N}, \sum_{n=1}^{m} |x_n| \le B$

-
$$orall m \in {f N}$$
, $\{1,\ldots,m\}$ is a finite set $\implies \exists k \in {f N}$ such that

$$\sigma(\{1,\ldots,m\}) \subseteq \{1,\ldots,k\},\$$

hence,

$$\sum_{n=1}^{m} |x_{\sigma(n)}| = \sum_{n \in \sigma(\{1, \dots, m\})} |x_n| \le \sum_{n=1}^{k} |x_n| \le B$$

 $\implies orall m \in \mathbf{N}$, $\sum_{n=1}^m |x_{\sigma(n)}|$ is bounded

• we now show that $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$, let $\epsilon > 0$ - $\sum_{n=1}^{\infty} x_n = x \implies \exists M_0 \in \mathbf{N}$ such that for all $k > m \ge M_0$, we have

$$\left|\sum_{n=1}^{m} x_n - x\right| < \epsilon/2$$
 and $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon/2$

– the set $\{1,\ldots,M_0\}$ is finite $\implies \exists M\in\mathbf{N},\ M>M_0$ such that

$$\{1,\ldots,M_0\}\subseteq\sigma(\{1,\ldots,M\})$$

hence, for all $m \geq M$, let $p = \max(\sigma(\{1,\ldots,m\})) > M_0$, we have

$$\sigma(\{1,\ldots,m\}) = \{1,\ldots,M_0\} \cup \{M_0+1,\ldots,p\}$$

– consider the partial sums of $\sum_{n=1}^\infty x_{\sigma(x)}$, for all $m\geq M$, we have

$$\left|\sum_{n=1}^{m} x_{\sigma(n)} - x\right| = \left|\sum_{n \in \sigma(\{1,\dots,m\})} x_n - x\right| = \left|\sum_{n=1}^{M_0} x_n - x + \sum_{n=M_0+1}^{p} x_n\right|$$
$$\leq \left|\sum_{n=1}^{M_0} x_n - x\right| + \left|\sum_{n=M_0+1}^{p} x_n\right| < \epsilon/2 + \epsilon/2 = \epsilon$$
$$\implies \lim_{m \to \infty} \sum_{n=1}^{m} x_{\sigma(n)} = x \implies \sum_{n=1}^{\infty} x_{\sigma(n)} = x$$

4-25

Series

5. Continuous functions

- cluster points of sets
- limits of functions and sequential properties
- left and right limits
- continuous functions
- operations that preserves continuity
- extreme value theorem
- intermediate value theorem
- uniform and Lipschitz continuity

Continuous functions

5-1

Cluster points of sets

Definition 5.1 Let $S \subseteq \mathbf{R}$. We say that the point $c \in \mathbf{R}$ is a **cluster point** of S if for all $\delta > 0$, we have $(c - \delta, c + \delta) \cap S \setminus \{c\} \neq \emptyset$, *i.e.*, for all $\delta > 0$, there exists some $x \in S$, such that $0 < |x - c| < \delta$.

examples:

- $S = \{1/n \mid n \in \mathbf{N}\}$ has a cluster point c = 0
- S = (0, 1) has a set of cluster points given by [0, 1]
- $S = \mathbf{Q}$ has a set of cluster points given by \mathbf{R}
- $S = \{0\}$ has no cluster points
- $S = \mathbf{Z}$ has no cluster points

Theorem 5.2 Let $S \subseteq \mathbf{R}$. Then c is a cluster point of S if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ of elements in $S \setminus \{c\}$ such that $\lim_{n \to \infty} x_n = c$.

proof:

• suppose c is a cluster point of S, then $\forall \delta > 0$, $\exists x \in S$ such that $0 < |x - c| < \delta$ - $\forall n \in \mathbb{N}$, choose $x_n \in S$ such that $0 < |x_n - c| < \frac{1}{n}$

 $-\frac{1}{n} \to 0 \implies |x_n - c| \to 0 \implies x_n \to c$

- suppose there exists a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in S \setminus \{c\}$ for all $n \in \mathbb{N}$ such that $x_n \to c$, let $\delta > 0$
 - $\ x_n \to c \text{ with } x_n \in S \setminus \{c\} \implies \exists M \in \mathbf{N} \text{ such that } \forall n \geq M, \ 0 < |x_n c| < \delta$
 - choose $x = x_M$, then we have $0 < |x c| < \delta \implies S$ has cluster point c

Continuous functions

Limits of functions

Definition 5.3 Let $f: S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose there exists an $L \in \mathbf{R}$, and for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in S$ and $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. We then say f(x) converges to L as x goes to c, and we write

$$f(x) \to L$$
 as $x \to c$.

We say L is a **limit** of f(x) as x goes to c, and if L is unique, we write

$$\lim_{x \to c} f(x) = L.$$

Remark 5.4 The function $f: S \to \mathbf{R}$ does not converge to $L \in \mathbf{R}$ as x goes to a cluster point c of S implies that there exists some $\epsilon > 0$, such that for all $\delta > 0$, there exists some $x \in S$ and $0 < |x - c| < \delta$, so that $|f(x) - L| \ge \epsilon$.

Theorem 5.5 Let $f: S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. If $f(x) \to L_1$ and $f(x) \to L_2$ as $x \to c$, then $L_1 = L_2$.

proof: let $\epsilon > 0$

- $f(x) \to L_1$ as $x \to c \implies \exists \delta_1 > 0$ such that for all $x \in S$ and $0 < |x c| < \delta_1$, $|f(x) L_1| < \epsilon/2$
- $f(x) \to L_2$ as $x \to c \implies \exists \delta_2 > 0$ such that for all $x \in S$ and $0 < |x c| < \delta_2$, $|f(x) L_2| < \epsilon/2$
- choose $\delta = \min\{\delta_1, \delta_2\}$, then for all $x \in S$ and $0 < |x c| < \delta$, we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \le |f(x) - L_1| + |f(x) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

$$\implies L_1 = L_2$$

Continuous functions

Example 5.6 Let f(x) = ax + b. Then, for all $c \in \mathbf{R}$, we have $\lim_{x\to c} f(x) = ac + b$.

proof: let $\epsilon > 0$, choose $\delta = \frac{\epsilon}{|a|+1}$, then for all $x \in \mathbf{R}$ and $0 < |x - c| < \delta$, we have

$$|f(x) - (ac+b)| = |(ax+b) - (ac+b)| = |a||x-c| < |a|\delta = \frac{|a|}{|a|+1}\epsilon \le \epsilon$$

Example 5.7 Let $f: (0, \infty) \to \mathbf{R}$ with $f(x) = \sqrt{x}$. Then, for all c > 0, we have $\lim_{x\to c} f(x) = \sqrt{c}$.

proof: let $\epsilon > 0$, choose $\delta = \epsilon \sqrt{c}$, then for all x > 0 and $0 < |x - c| < \delta$, we have

$$|f(x) - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| = \left|\frac{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})}{\sqrt{x} + \sqrt{c}}\right| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| \le \frac{|x - c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} < \epsilon$$

Example 5.8 Let $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$. Then, $\lim_{x \to 0} f(x) = 1 \ (\neq f(0))$.

proof: let $\epsilon > 0$, choose $\delta = 1$, then $\forall x$ satisfies $0 < |x| < \delta$, we have $x \neq 0 \implies \forall x$ satisfies $0 < |x| < \delta$, we have $|f(x) - 1| = |1 - 1| = 0 < \epsilon$

Continuous functions

Theorem 5.9 Let $f: S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Then, the following statements are equivalent:

- The function f(x) converges to $L \in \mathbf{R}$ as x goes to c, *i.e.*, $\lim_{x\to c} f(x) = L$.
- For all sequences $(x_n)_{n=1}^{\infty}$ in $S \setminus \{c\}$ such that $\lim_{n \to \infty} x_n = c$, we have $\lim_{n \to \infty} f(x_n) = L$.

proof:

- suppose $\lim_{x\to c} f(x) = L$, let $\epsilon > 0$
 - $\exists \delta > 0$, such that for all $x \in S$ and $0 < |x c| < \delta$, we have $|f(x) L| < \epsilon$
 - $x_n \to c$, $x_n \in S \setminus \{c\} \implies \exists M \in \mathbf{N}$ such that $\forall n \ge M$, $0 < |x_n c| < \delta \implies \forall n \ge M$, we have $|f(x_n) L| < \epsilon$, *i.e.*, $f(x_n) \to L$
- suppose for all sequences in $S \setminus \{c\}$ s.t. $x_n \to c$, we have $f(x_n) \to L$
 - assume $\lim_{x\to c} f(x) \neq L \implies \exists \epsilon > 0$ s.t. $\forall \delta > 0$, there exists some $x \in S$ and $0 < |x c| < \delta$, so that $|f(x) L| \geq \epsilon$
 - choose a sequence $(x_n)_{n=1}^{\infty}$ s.t. $\forall n \in \mathbb{N}$, $x_n \in S \setminus \{c\}$, $0 < |x_n c| < \frac{1}{n}$, and $|f(x_n) L| \ge \epsilon$ for all $n \in \mathbb{N}$
 - however, $\frac{1}{n} \to 0 \implies x_n \to c \implies f(x_n) \to L \implies \exists M \in \mathbf{N} \text{ s.t. } \forall n \ge M$, $|f(x_n) L| < \epsilon$, which is a contradiction

Continuous functions

5-7

Theorem 5.10 For all $c \in \mathbf{R}$, we have $\lim_{x\to c} x^2 = c^2$.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R} \setminus \{c\}$ such that $x_n \to c$, then according to theorem 3.24, we have $x_n^2 \to c^2 \implies \lim_{x \to c} x^2 = c^2$ (theorem 5.9)

Theorem 5.11 The limit $\lim_{x\to 0} \sin(1/x)$ does not exist, but $\lim_{x\to 0} x \sin(1/x) = 0$.

proof:

- we first show that $\lim_{x\to 0} x \sin(1/x) = 0$: let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R} \setminus \{0\}$ such that $x_n \to 0$; since $0 \le |x_n \sin(1/x_n)| \le |x_n|$ for all $n \in \mathbf{N}$, and $x_n \to 0$, we have $|x_n \sin(1/x_n)| \to 0 \implies \lim_{x\to 0} x \sin(1/x) = 0$
- we now show that $\lim_{x\to 0} \sin(1/x)$ does not exist:
 - choose a sequence $(x_n)_{n=1}^\infty$ where $x_n=\frac{2}{(2n-1)\pi},$ then we have $x_n\to 0$
 - consider the sequence $(\sin(1/x_n))_{n=1}^{\infty}$, we have

$$\sin(1/x_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n+1}$$

 $\implies (\sin(1/x_n))_{n=1}^{\infty}$ does not converge $\implies \lim_{x \to 0} \sin(1/x)$ does not exist

Theorem 5.12 Let $f, g: S \to \mathbf{R}$ be functions and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose $f(x) \leq g(x)$ for all $x \in S$, and we have $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence in $S \setminus \{c\}$ such that $x_n \to c$

- $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist $\implies (f(x_n))_{n=1}^{\infty}$ and $(g(x_n))_{n=1}^{\infty}$ converges
- let $f(x_n) \to L_1$, $g(x_n) \to L_2$, since $f(x) \le g(x)$ for all $x \in S$, we have $L_1 \le L_2$, *i.e.*, $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$

similarly, we can prove the following theorems using the properties of sequences:

Theorem 5.13 Let $f: S \to \mathbf{R}$ be a function and c be a cluster point of $S \subseteq \mathbf{R}$. Suppose the limit $\lim_{x\to c} f(x)$ exists, and there exists $a, b \in \mathbf{R}$ such that $a \leq f(x) \leq b$ for all $x \in S \setminus \{c\}$, then $a \leq \lim_{x\to c} f(x) \leq b$.

Continuous functions

Theorem 5.14 Let c be a cluster point of $S \subseteq \mathbf{R}$, and $f, g, h: S \to \mathbf{R}$ be functions such that $f(x) \leq g(x) \leq h(x)$ for all $x \in S \setminus \{c\}$. Suppose $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$.

Theorem 5.15 Let c be a cluster point of $S \subseteq \mathbf{R}$, and $f, g: S \to \mathbf{R}$ be functions such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, we have:

- $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x);$
- $\lim_{x\to c} (f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x);$
- if $\lim_{x\to c} g(x) \neq 0$ and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)};$$

Theorem 5.16 Let c be a cluster point of $S \subseteq \mathbf{R}$ and $f: S \to \mathbf{R}$ be a function such that $\lim_{x\to c} f(x)$ exists, then we have $\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)|$.

Definition 5.17 Let $S \subseteq \mathbf{R}$ and $f \colon S \to \mathbf{R}$ be a function.

Suppose c is a cluster point of $S \cap (-\infty, c)$, we say f(x) converges to L as $x \to c^-$, if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $c - \delta < x < c$, we have $|f(x) - L| < \epsilon$. We call such a limit the **left limit** of f at c, denoted $\lim_{x\to c^-} f(x)$. Suppose c is a cluster point of $S \cap (c, \infty)$, we say f(x) converges to L as $x \to c^+$, if

for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $c < x < c + \delta$, we have $|f(x) - L| < \epsilon$. We call such a limit the **right limit** of f at c, denoted $\lim_{x\to c^+} f(x)$.

Example 5.18 Consider the function f given by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0, \end{cases}$$

we have $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = 1$, even if f(0) is undefined.

Continuous functions

Continuous functions

Definition 5.19 Let $S \subseteq \mathbf{R}$ and $c \in S$. We say the function f is **continuous** at c if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

We say the function f is continuous on the set U for $U \subseteq S$ if f is continuous at every point of U.

Remark 5.20 The function f is not continuous at point $c \in S$ if there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exists some $x \in S$ and $|x - c| < \delta$, so that $|f(x) - f(c)| \ge \epsilon$.

Example 5.21 The function f(x) = ax + b is continuous on **R**.

proof: let $c \in \mathbf{R}$, $\epsilon > 0$, choose $\delta = \frac{\epsilon}{|a|+1}$, then for all $x \in \mathbf{R}$ and $|x - c| < \delta$, we have $|f(x) - f(c)| = |ax + b - ac - b| = |a||x - c| < |a|\delta = \frac{|a|}{|a|+1}\epsilon \le \epsilon$

Example 5.22 The function *f* given by

$$f(x) = \begin{cases} 1 & x \neq 0\\ 2 & x = 0 \end{cases}$$

is not continuous at c = 0.

proof: choose $\epsilon = 1$ and let $\delta > 0$, then $x = \delta/2$ satisfies $|x| < \delta$, but

$$|f(x) - f(0)| = |1 - 0| = 1 \ge \epsilon$$

Continuous functions

Theorem 5.23 Let $S \subseteq \mathbf{R}$ be a set, $c \in S$ be a point, and $f \colon S \to \mathbf{R}$ be a function.

- If c is not a cluster point of S, then the function f is continuous at c.
- If c is a cluster point of S, then the function f is continuous at c if and only if lim_{x→c} f(x) = f(c).
- The function f is continuous at c if and only if for all sequences $(x_n)_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = f(c)$.

proof: to show the first statement, let $\epsilon > 0$

- $c \in S$ and c is not a cluster point of $S \implies \exists \delta > 0$ s.t. $(c \delta, c + \delta) \cap S = \{c\}$
- then for all $x \in S$ such that $|x c| < \delta$, we have x = c, and hence,

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

we now show the second statement:

- suppose f is continuous at c, let $\epsilon > 0$
 - f is continuous at $c\implies \exists \delta>0$ such that for all $x\in S$ and $|x-c|<\delta,$ we have $|f(x)-f(c)|<\epsilon$

- then $\forall x \in S \text{ s.t. } 0 < |x - c| < \delta$, $|f(x) - f(c)| < \epsilon \implies \lim_{x \to c} f(x) = f(c)$

• suppose $\lim_{x\to c} f(x) = f(c)$, let $\epsilon > 0$

- $f(x) \to f(c)$ as $x \to c \implies \exists \delta > 0$ such that for all $x \in S$ and $0 < |x c| < \delta$, we have $|f(x) f(c)| < \epsilon$
- then for all $x \in S$ such that $|x c| < \delta$: if x = c, we have

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

if $x \neq c$, we have $0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$

– put together, we conclude that the function f is continuous at c

we now show the third statement

• suppose f is continuous at c, let $(x_n)_{n=1}^{\infty}$ be a sequence in S, $x_n \to c$, let $\epsilon > 0$ - f is continuous at $c \implies \exists \delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$

$$- x_n \to c \implies \exists M \in \mathbf{N} \text{ such that } \forall n \ge M, \ |x_n - c| < \delta \implies \forall n \ge M, \ |f(x_n) - f(c)| < \epsilon \implies (f(x_n))_{n=1}^{\infty} \to f(c)$$

- suppose for all $(x_n)_{n=1}^{\infty}$ in S such that $x_n \to c$, we have $f(x_n) \to f(c)$
 - assume f is not continuous at $c \implies \exists \epsilon > 0, \forall \delta > 0, \exists x \in S$ such that $|x c| < \delta$, but $|f(x) f(c)| \ge \epsilon$

- choose
$$x_n \in S$$
 such that $\forall n \in \mathbf{N}, 0 \leq |x_n - c| < \frac{1}{n}$ but $|f(x_n) - f(x)| \geq \epsilon$

$$-\frac{1}{n} \to 0 \implies x_n \to c \implies f(x_n) \to f(c) \implies \exists M \in \mathbf{N} \text{ such that } \forall n \ge M,$$

 $|f(x_n) - f(c)| < \epsilon$, which is a contradiction

Continuous functions

TI	1eorem 5.24	· Tł	ne f	functions	$\sin x$	and	$\cos x$	are	continuous	functi	ons	on	R
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proof:

• recall the following properties of $\sin x$ and $\cos x$ for all $x \in \mathbf{R}$:

$$-\sin^2(x) + \cos^2(x) = 1 \implies |\sin x| \le 1 \text{ and } |\cos x| \le 1$$

$$-|\sin x| \le |x|$$

$$-\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

$$-\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

• we first show that $\sin x$ is continuous, let $c \in \mathbf{R}$, let $\epsilon > 0$, choose $\delta = \epsilon$, then for all $x \in \mathbf{R}$ such that $|x - c| < \delta$, we have

$$\left|\sin x - \sin c\right| = \left|2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)\right| \le 2\left|\sin\left(\frac{x-c}{2}\right)\right| \le 2\frac{|x-c|}{2} = |x-c| < \epsilon$$

• we now show that $\cos x$ is continuous, let $c \in \mathbf{R}$, let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n \to c$, then we have $x_n + \frac{\pi}{2} \to c + \frac{\pi}{2}$, and hence,

$$\lim_{n \to \infty} \cos x_n = \lim_{n \to \infty} \sin \left(x_n + \frac{\pi}{2} \right) = \sin \left(c + \frac{\pi}{2} \right) = \cos c$$

Theorem 5.25 Dirichlet function. The Dirichlet function given by

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

is not continuous on all of \mathbf{R} .

proof: let $c \in \mathbf{R}$

• if $c \in \mathbf{Q}$, then for all $n \in \mathbf{N}$, there exists $x_n \notin \mathbf{Q}$ such that $c < x_n < c + \frac{1}{n}$; $\frac{1}{n} \to 0 \implies x_n \to c$, however,

$$\lim_{n \to \infty} f(x_n) = 0 \neq f(c) = 1$$

 $\implies (f(x_n))_{n=1}^\infty$ does not converge to f(c)

• if $c \notin \mathbf{Q}$, then for all $n \in \mathbf{N}$, there exists $x_n \in \mathbf{Q}$ such that $c < x_n < c + \frac{1}{n}$; $\frac{1}{n} \to 0 \implies x_n \to c$, however,

$$\lim_{n \to \infty} f(x_n) = 1 \neq f(c) = 0$$

$$\implies (f(x_n))_{n=1}^{\infty}$$
 does not converge to $f(c)$

Continuous functions

Operations that preserves continuity

Theorem 5.26 Let $f, g: S \to \mathbf{R}$ be functions on $S \subseteq \mathbf{R}$ and are continuous at $c \in S$.

- The function f + g is continuous at c.
- The function $f \cdot g$ is continuous at c.
- If $g(x) \neq 0$ for all $x \in S$, then the function f/g is continuous at c.

proof: we show that the function f + g is continuous at c, the other two statements can be proved similarly; let $(x_n)_{n=1}^{\infty}$ be a sequence in S with $x_n \to c$

- f is continuous at $c \implies \lim_{n \to \infty} f(x_n) = f(c)$
- g is continuous at $c \implies \lim_{n \to \infty} g(x_n) = g(c)$
- hence, $\lim_{n\to\infty}(f(x_n) + g(x_n)) = f(c) + g(c) \implies f + g$ is continuous at c

Theorem 5.27 Let $f: B \to \mathbf{R}$ and $g: A \to B$ be functions on $A, B \subseteq \mathbf{R}$. If g is continuous at $c \in A$ and f is continuous at $g(c) \in B$, then $f \circ g$ is continuous at c.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence in A and $x_n \to c \implies g(x_n) \to g(c) \implies f(g(x_n)) \to f(g(c)) \implies f \circ g$ is continuous at c

Continuous functions

Theorem 5.28 Let *f* be a polynomial function of the form

$$f(x) = a_p x^p + \dots + a_1 x + a_0.$$

Then, the function f is continuous on **R**.

proof: let $c \in \mathbf{R}$, let $(x_n)_{n=1}^{\infty}$ be a sequence in \mathbf{R} and $x_n \to c$, then we have

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} (a_p x_n^p + \dots + a_1 x_n + a_0)$$
$$= a_p \lim_{n \to \infty} x_n^p + \dots + a_1 \lim_{n \to \infty} x_n + a_0$$
$$= a_p c^p + \dots + a_1 c + a_0 = f(c)$$

Example 5.29 Theorems 5.26 and 5.27 allows us to show that some given function is continuous without a huge $\epsilon - \delta$ proof, for example:

- The function $1/x^2$ is continuous on $(0,\infty)$, since x^2 is continuous on $(0,\infty)$.
- The function $(\cos(1/x^2))^2$ is continuous on $(0, \infty)$, since $\cos x$ is continuous on \mathbf{R} , and x^2 is continuous on $(0, \infty)$.

Continuous functions

Extreme value theorem

Definition 5.30 A function $f: S \to \mathbf{R}$ is **bounded** if there exists some $B \ge 0$ such that for all $x \in S$, we have $|f(x)| \le B$.

Theorem 5.31 If the function $f: [a, b] \rightarrow \mathbf{R}$ is continuous then f is bounded.

proof:

- suppose f is unbounded, then $\forall B \ge 0$, $\exists x \in [a, b]$ such that |f(x)| > B
- let $(x_n)_{n=1}^\infty$ be a sequence in [a,b] such that for all $n \in \mathbf{N}$, $|f(x_n)| > n$
- $(x_n)_{n=1}^{\infty}$ is in $[a, b] \implies (x_n)_{n=1}^{\infty}$ is bounded \implies there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ (theorem 3.37) that converges to $c \in \mathbf{R}$
- $a \le x_n \le b \implies a \le x_{n_i} \le b \implies c \in [a, b]$
- f is continuous on $[a,b] \implies f(x_{n_i}) \to f(c) \implies (f(x_{n_i}))_{i=1}^{\infty}$ is bounded
- however, $|f(x_{n_i})|>n_i\implies (n_i)_{i=1}^\infty$ is bounded, which is a contradiction

Definition 5.32 Let $f: S \to \mathbf{R}$ be a function. We say the function f achieves an **absolute minimum** at c if $f(x) \ge f(c)$ for all $x \in S$. We say the function f achieves an **absolute maximum** at d if $f(x) \le f(d)$ for all $x \in S$.

Theorem 5.33 *Extreme value theorem.* Let $f: [a, b] \to \mathbf{R}$ be a function on a closed, bounded interval [a, b]. If the function f is continuous on [a, b], then f achieves absolute maximum and absolute minimum on [a, b].

proof: we show the case for absolute maximum

- f is continuous on $[a, b] \implies f$ is bounded \implies the set $E = \{f(x) \mid x \in [a, b]\}$ is bounded $\implies \sup E \in \mathbf{R}$ exists
- $\sup E$ is the supremum of $\{f(x) \mid x \in [a,b]\} \implies \forall x \in [a,b], f(x) \leq \sup E$, and, there exists some sequence $(f(x_n))_{n=1}^{\infty}$ with $x_n \in [a,b]$ such that $f(x_n) \to \sup E$
- $(x_n)_{n=1}^{\infty}$ is in $[a, b] \implies$ there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ such that $x_{n_i} \to d$ and $d \in [a, b] \implies f(x_{n_i}) \to f(d)$ (since f is continuous)
- $f(x_n) \to \sup E \implies f(x_{n_i}) \to \sup E \implies \sup E = f(d) \implies$ there exists a point $d \in [a, b]$ such that $f(x) \le f(d)$ for all $x \in [a, b]$

Continuous functions

Remark 5.34 To apply the extreme value theorem, the function f has to be continuous on a closed, bounded interval.

If the function $f: [a, b] \to \mathbf{R}$ is not continuous, consider the function given by

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \text{ or } x = 1\\ x & x \in (0, 1), \end{cases}$$

which neither achieves an absolute maximum nor an absolute minimum on [0, 1].

If the function $f: S \to \mathbf{R}$ is continuous but S not closed and bounded, consider the function given by

$$f(x) = \frac{1}{x} - \frac{1}{1-x}, \quad S = (0,1),$$

which neither achieves an absolute maximum nor an absolute minimum on [0, 1].

5-	2	1

Theorem 5.35 Let $f: [a,b] \to \mathbf{R}$ be a continuous function. If f(a) < 0 and f(b) > 0, then there exists some $c \in (a,b)$ such that f(c) = 0.

proof: let $a_1 = a$, $b_1 = b$, for all $n \in \mathbb{N}$, given a_n and b_n , define a_{n+1} and b_{n+1} as:

- $a_{n+1} = a_n$, $b_{n+1} = \frac{a_n + b_n}{2}$, if $f\left(\frac{a_n + b_n}{2}\right) \ge 0$
- $a_{n+1} = \frac{a_n + b_n}{2}$, $b_{n+1} = b_n$, if $f\left(\frac{a_n + b_n}{2}\right) < 0$

then the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ has the following properties:

- $a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$ for all $n \in \mathbb{N} \implies (a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are monotone and bounded $\implies (a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge, let $a_n \to c$, $b_n \to d$
- $f(a_n) \leq 0$, $f(b_n) \geq 0$ for all $n \in \mathbf{N}$, since f is continuous, $c, d \in [a, b] \implies \lim_{n \to \infty} f(a_n) = f(c) \leq 0$ and $\lim_{n \to \infty} f(b_n) = f(d) \geq 0$
- $b_{n+1} a_{n+1} = \frac{b_n a_n}{2} = \frac{b_{n-1} a_{n-1}}{2^2} = \dots = \frac{b-a}{2^n} \implies b_n a_n = \frac{1}{2^{n-1}}(b-a)$ $\implies \lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty} \frac{1}{2^{n-1}}(b-a) = 0 = \lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n$ $\implies \lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n \implies c = d$

put together, we have $f(c) \leq 0$, $f(d) \geq 0$, and $f(c) = f(d) \implies f(c) = f(d) = 0$ $\implies \exists c \in (a, b)$ such that f(c) = 0

Continuous functions

Theorem 5.36 Bolzano's intermediate value theorem. Let $f: [a,b] \to \mathbf{R}$ be a continuous function. Suppose $y \in \mathbf{R}$ such that f(a) < y < f(b) or f(b) < y < f(a), then there exists a $c \in (a,b)$ such that f(c) = y.

proof: we consider the case for f(a) < y < f(b), the other case is similar

- let $g: [a, b] \to \mathbf{R}$ be a function given by g(x) = f(x) y, then g is continuous on [a, b] (theorem 5.26)
- $f(a) < y < f(b) \implies g(a) = f(a) y < 0, \ g(b) = f(b) y > 0 \implies \exists c \in (a, b)$ such that g(c) = f(c) - y = 0 (theorem 5.35) $\implies \exists c \in (a, b)$ such that f(c) = y

Theorem 5.37 Let $f: [a, b] \to \mathbf{R}$ be a continuous function. Suppose the function f achieves absolute minimum at $c \in [a, b]$, and achieves absolute maximum at $d \in [a, b]$. Then, we have f([a, b]) = [f(c), f(d)], *i.e.*, every value between the absolute minimum value and the absolute maximum value is achieved.

proof:

- according to theorem 5.33, we have $f([a,b]) \subseteq [f(c), f(d)]$
- according to theorem 5.36, we have $[f(c), f(d)] \subseteq f([c, d]) \subseteq f([a, b])$
- hence, f([a, b]) = [f(c), f(d)]

Remark 5.38 Similarly, theorem 5.36 is false if the function f is not continuous.

Example 5.39 The polynomial given by $f(x) = x^{2021} + x^{2020} + 9.03x + 1$ has at least one real root.

proof: we have f(0) = 1 > 0 and f(-1) = -8.03 < 0, hence, by theorem 5.36, there exists some $c \in (-1,0)$ such that f(c) = 0

Continuous functions

Uniform continuity

Example 5.40 The function $f(x) = \frac{1}{x}$ is continuous on (0, 1).

proof: let $c \in (0,1)$ and $\epsilon > 0$, choose $\delta = \min\left\{\frac{c}{2}, \frac{c^2}{2}\epsilon\right\}$, then $\forall x \in (0,1)$ such that $|x-c| < \delta$, we have

- $||x| |c|| \le |x c| < \delta \le \frac{c}{2} \implies -\frac{c}{2} < |x| c \implies \frac{1}{|x|} < \frac{2}{c}$
- hence, $\left|\frac{1}{x} \frac{1}{c}\right| = \frac{|x-c|}{|x|c} < \frac{\delta}{|x|c} < \frac{2\delta}{c^2} \le \frac{2}{c^2} \cdot \frac{c^2}{2}\epsilon = \epsilon$

Remark 5.41 Example 5.40 shows that in the definition of function continuity, the number δ can depend on both the number ϵ and the point c.

Definition 5.42 Let $f: S \to \mathbf{R}$ be a function. We say the function f is **uniformly** continuous on S if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x, c \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

Remark 5.43 In the definition of uniform continuity, the number δ only depends on ϵ .

Example 5.44 The function $f(x) = x^2$ is uniformly continuous on [0, 1].

proof: let $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$, then for all $x, c \in [0, 1]$ and $|x - c| < \delta$, we have $|x + c| \le 2$, and hence,

$$|f(x) - f(c)| = |x^2 - c^2| = |x + c||x - c| < |x + c|\delta \le 2\delta = 2 \cdot \epsilon = \epsilon$$

Remark 5.45 Let $f: S \to \mathbf{R}$ be a function. We say the function f is not uniformly continuous on S if there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exists some $x, c \in S$ and $|x - c| < \delta$ so that $|f(x) - f(c)| \ge \epsilon$.

Example 5.46 The function $f(x) = \frac{1}{x}$ is not uniformly continuous on (0,1).

- **proof:** choose $\epsilon = 2$, let $\delta > 0$, choose $c = \min \{\delta, \frac{1}{2}\}$, $x = \frac{c}{2}$, then we have
 - $x, c \in (0, 1)$ and $|x c| = \frac{c}{2} \le \frac{\delta}{2} < \delta$
 - $\left|\frac{1}{x} \frac{1}{c}\right| = \frac{|x-c|}{|x||c|} = \frac{c}{2} \cdot \frac{2}{c^2} = \frac{1}{c} \ge 2 = \epsilon$

Continuous functions

5-27

Example 5.47 The function given by $f(x) = x^2$ is not uniformly continuous on **R**.

proof: choose $\epsilon = 2$, let $\delta > 0$, choose $c = \frac{2}{\delta}$, $x = c + \frac{\delta}{2}$, then we have

- $x, c \in \mathbf{R}$ and $|x c| = \frac{\delta}{2} < \delta$
- $|x^2 c^2| = |x + c||x c| = (2c + \frac{\delta}{2}) \cdot \frac{\delta}{2} = (\frac{4}{\delta} + \frac{\delta}{2}) \cdot \frac{\delta}{2} = 2 + \frac{\delta^2}{4} \ge 2 = \epsilon$

Theorem 5.48 Let $f: [a, b] \to \mathbf{R}$ be a function. Then, the function f is continuous on [a, b] if and only if f is uniformly continuous on [a, b].

proof:

- suppose f is uniformly continuous on [a, b]: let c ∈ [a, b], ε > 0, then according to uniform continuity, ∃δ > 0 such that for all x ∈ [a, b] and |x c| < δ, we have |f(x) f(c)| < ε
- suppose f is continuous on [a, b]
 - assume f is not uniformly continuous on [a, b], then $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exists $x, c \in [a, b]$ such that $|x c| < \delta$ but $|f(x) f(c)| \ge \epsilon$

- choose sequences $(x_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ such that for all $n \in \mathbb{N}$, $x_n, c_n \in [a, b]$, $|x_n c_n| < \frac{1}{n}$, but $|f(x_n) f(c_n)| \ge \epsilon$
- since $x_n \in [a, b]$ for all $n \in \mathbb{N}$, there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $x_{n_i} \to c$ and $c \in [a, b]$ (theorem 3.37)
- take subsequence $(c_{n_i})_{i=1}^{\infty}$ of $(c_n)_{n=1}^{\infty}$ according to the indexes n_i of $(x_{n_i})_{i=1}^{\infty}$, then $c_{n_i} \in [a, b]$ for all $n \in \mathbb{N}$ \implies there exists a subsequence $(c_{n_{i_j}})_{j=1}^{\infty}$ such that $c_{n_{i_j}} \rightarrow d$ and $d \in [a, b]$
- take subsequence $(x_{n_{i_j}})_{j=1}^{\infty}$ of $(x_{n_i})_{i=1}^{\infty}$ according to the indexes n_{i_j} of $(c_{n_{i_j}})_{j=1}^{\infty}$, then $x_{n_{i_j}} \to c$ since $x_{n_i} \to c$
- $\begin{array}{l} \ 0 \leq |x_{n_{i_j}} c_{n_{i_j}}| < \frac{1}{n_{i_j}} \text{ and } \frac{1}{n_{i_j}} \to 0 \implies \lim_{j \to \infty} |x_{n_{i_j}} c_{n_{i_j}}| = 0 \implies \lim_{j \to \infty} x_{n_{i_j}} = \lim_{j \to \infty} c_{n_{i_j}} \implies c = d \end{array}$
- since f is continuous on [a,b] and $x_{n_{i_i}} \rightarrow c$, $c_{n_{i_i}} \rightarrow c$, we have

$$\lim_{j \to \infty} f(x_{n_{i_j}}) = \lim_{j \to \infty} f(c_{n_{i_j}}) = f(c)$$
$$\implies \quad 0 = |f(c) - f(c)| = \lim_{j \to \infty} |f(x_{n_{i_j}}) - f(c_{n_{i_j}})| \ge \epsilon,$$

which is a contradiction

Continuous functions

Lipschitz continuity

Definition 5.49 Let $f: S \to \mathbf{R}$ be a function. We say the function f is **Lipschitz** continuous on S if there exists some $K \ge 0$ such that for all $x, y \in S$, we have $|f(x) - f(y)| \le K|x - y|$.

Remark 5.50 Geometrically, the function f is Lipschitz continuous if and only if all lines intersects the graph of f in at least two distinct points has slope in absolute value less than or equal to K.

Theorem 5.51 Let $f: S \to \mathbf{R}$ be a function. If the function f is Lipschitz continuous, then f is uniformly continuous.

proof: let $\epsilon > 0$

- f is Lipschitz continuous $\implies \exists K \ge 0$ such that for all $x, y \in S$, we have $|f(x) f(y)| \le K|x y|$
- choose $\delta = \epsilon/(K+1)$, then for all $x, y \in S$ and $|x-y| < \delta$, we have

$$|f(x) - f(y)| \le K|x - y| < K\delta = \frac{K}{K+1}\epsilon < \epsilon$$

Continuous functions

Example 5.52 The function $f(x) = \sqrt{x}$ is Lipschitz continuous on $[1, \infty)$, but is not Lipschitz continuous on $[0, \infty)$.

proof:

• consider the function $f: [1, \infty) \to \mathbf{R}$ given by $f(x) = \sqrt{x}$, then $\forall x, y \in [1, \infty)$: - $x \ge 1, y \ge 1 \implies \sqrt{x} + \sqrt{y} \ge 2$

- hence,

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \le \frac{1}{2}|x - y|$$

 $\implies f$ is Lipschitz continuous with K=1/2

• consider the function $g\colon [0,\infty)\to {\bf R}$ given by $g(x)=\sqrt{x}$, let $K\ge 0$, choose $x=0,\ y=\frac{1}{K^2+1}$, then

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \left|\frac{\sqrt{x} - \sqrt{y}}{x - y}\right| = \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} = \sqrt{K^2 + 1} > \sqrt{K^2} = K$$
$$\implies |f(x) - f(y)| > K|x - y|$$

Continuous functions

6. Derivative

- definition and basic properties
- differentiation rules
- Rolle's theorem and mean value theorem
- Taylor's theorem

Derivative

Derivative of functions

Definition 6.1 Let I be an interval, let $f: I \to \mathbf{R}$ be a function, and let $c \in I$. We say the function f is **differentiable** at c if the limit

$$L = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. We call L the **derivative** of f at c, and we write f'(c) = L.

If f is differentiable at all $c \in I$, then we say the function f is differentiable, and we write f' or $\frac{df}{dx}$ for the function f'(x), $x \in I$.

Example 6.2 Consider the function f(x) = ax + b, then f'(c) = a for all $c \in \mathbf{R}$.

proof: let $x, c \in \mathbf{R}$, then we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{ax + b - (ac + b)}{x - c} = \lim_{x \to c} \frac{a(x - c)}{x - c} = \lim_{x \to c} a = a$$

Example 6.3 Consider the function $f(x) = x^2$, then f'(c) = 2c for all $c \in \mathbf{R}$.

proof: let $x, c \in \mathbf{R}$, then we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} \frac{(x + c)(x - c)}{x - c} = \lim_{x \to c} (x + c) = 2c$$

Theorem 6.4 Suppose the function $f: I \to \mathbf{R}$ is differentiable at $c \in I$, then f is continuous at c.

proof: f is differentiable at $c \in I \implies$ the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, hence,

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} (x - c) + f(c) \right) = f'(c) \cdot 0 + f(c) = f(c)$$

Remark 6.5 The converse of theorem 6.4 does not hold.

Derivative

Example 6.6 The function f(x) = |x| is not differentiable at 0.

proof: let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n = \frac{(-1)^n}{n}$ for all $n \in \mathbb{N}$ • $0 \le \left|\frac{(-1)^n}{n}\right| \le \frac{1}{n}$ and $\frac{1}{n} \to 0 \implies x_n \to 0$ • consider the sequence $\left(\frac{f(x_n) - f(0)}{x_n - 0}\right)_{n=1}^{\infty}$, we have

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{|x_n|}{x_n} = \frac{\left|\frac{(-1)^n}{n}\right|}{\frac{(-1)^n}{n}} = (-1)^n$$

• $\lim_{n\to\infty} (-1)^n$ does not exist $\implies \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist

Remark 6.7 There exist functions that are continuous but nowhere differentiable.

Theorem 6.8 Let I be an interval, let $f: I \to \mathbf{R}$ and $g: I \to \mathbf{R}$ be differentiable functions at $c \in I$.

- Linearity. Let $\alpha \in \mathbf{R}$. Define $h(x) = \alpha f(x) + g(x)$, then $h'(c) = \alpha f'(c) + g'(c)$.
- Product rule. Define h(x) = f(x)g(x), then h'(c) = f'(c)g(c) + f(c)g'(c).
- Quotient rule. If $g(x) \neq 0$ for all $x \in I$, define h(x) = f(x)/g(x), then

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

proof: f, g differentiable at $c \implies \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, $\lim_{x \to c} \frac{g(x) - g(c)}{x - c}$ exists, and f, g continuous at $c \implies \lim_{x \to c} f(x) = f(c)$, $\lim_{x \to c} g(x) = g(c)$

• if $h(x) = \alpha f(x) + g(c)$, then we have

$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \frac{\alpha f(x) + g(x) - \alpha f(c) - g(c)}{x - c}$$
$$= \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \alpha f'(c) + g'(c)$$

Derivative

•	if	h((x)	=	f	f(x)	g	(x),	then	we	have
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$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c}$$
$$= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c) + f(x)g(c) - f(x)g(c)}{x - c}$$
$$= \lim_{x \to c} \frac{g(c)(f(x) - f(c)) + f(x)(g(x) - g(c))}{x - c}$$
$$= g(c) \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} f(x) \frac{g(x) - g(c)}{x - c} = f'(c)g(c) + f(c)g'(c)$$

• if h(x) = f(x)/g(x), then we have

$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \frac{f(x)/g(x) - f(c)/g(c)}{x - c} = \lim_{x \to c} \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x)}{x - c}$$
$$= \lim_{x \to c} \frac{1}{g(x)g(c)} \frac{f(x)g(c) - f(c)g(x) + f(x)g(x) - f(x)g(x)}{x - c}$$
$$= \lim_{x \to c} \frac{1}{g(x)g(c)} \frac{g(x)(f(x) - f(c)) - f(x)(g(x) - g(c)))}{x - c}$$
$$= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

Derivative

Theorem 6.9 Chain rule. Let I_1 , I_2 be two intervals. Let $g: I_1 \to \mathbf{R}$ be differentiable at $c \in I_1$ and $f: I_2 \to \mathbf{R}$ be differentiable at g(c). Define $h: I_1 \to \mathbf{R}$ by $h = f \circ g$, then h is differentiable at c, and

$$h'(c) = f'(g(c))g'(c).$$

proof: let d = g(c)

• define the following functions:

$$u(y) = \begin{cases} \frac{f(y) - f(d)}{y - d} & y \neq d \\ f'(d) & y = d \end{cases} \quad \text{and} \quad v(x) = \begin{cases} \frac{g(x) - g(c)}{x - c} & x \neq c \\ g'(c) & x = c, \end{cases}$$

then we have

$$\lim_{y \to d} u(y) = \lim_{y \to d} \frac{f(y) - f(d)}{y - d} = f'(d) = u(d)$$
$$\lim_{x \to c} v(x) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c) = v(c),$$

i.e., u is continuous at d, v is continuous at c

Derivative

•	note that	f(y) -	f(d) =	$u(y)(y - e^{-y})$	d) and	g(x) - d	=v(x)(x -	- $c)$, we	have

$$h(x) - h(c) = f(g(x)) - f(d) = u(g(x))(g(x) - d) = u(g(x))v(x)(x - c)$$

• put together, we have

$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} u(g(x))v(x) = u(g(c))v(c) = f'(g(c))g'(c)$$
Definition 6.10 Let $f: S \to \mathbf{R}$ with $S \subseteq \mathbf{R}$.

The function f is said to have a **relative maximum** at $c \in S$ if there exists some $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $f(x) \le f(c)$.

The function f is said to have a **relative minimum** at $c \in S$ if there exists some $\delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $f(x) \ge f(c)$.

Theorem 6.11 If the function $f: [a, b] \to \mathbf{R}$ has a relative maximum or minimum at $c \in (a, b)$ and f is differentiable at c, then f'(c) = 0.

proof: we show the case for c being a relative maximum point

- $c \in (a, b)$ is an relative maximum point $\implies \exists \delta > 0$ such that for all $x \in [a, b]$ and $|x - c| < \delta$, we have $f(x) \le f(c)$
- let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n = c \frac{\delta}{2n}$ for all $n \in \mathbb{N}$, then we have $x_n < c$, $x_n \to c$, and $|x_n c| < \delta$ for all $n \in \mathbb{N} \implies f'(c) = \lim_{n \to \infty} \frac{f(x_n) f(c)}{x_n c} \ge 0$
- let $(y_n)_{n=1}^{\infty}$ be a sequence with $y_n = c + \frac{\delta}{2n}$ for all $n \in \mathbb{N}$, then we have $y_n > c$, $y_n \to c$, and $|y_n c| < \delta$ for all $n \in \mathbb{N} \implies f'(c) = \lim_{n \to \infty} \frac{f(y_n) f(c)}{y_n c} \le 0$

Derivative

Remark 6.12 In theorem 6.11, the function f does not necessarily have to be defined on a closed interval, but the point c where the relative extremum is achieved has to be on the open interval (a, b).

Remark 6.13 Absolute extremum is a special case of relative extremum.

Theorem 6.14 Rolle. Let the function $f: [a, b] \to \mathbf{R}$ be continuous and differentiable on (a, b). If f(a) = f(b), then there exists some $c \in (a, b)$ such that f'(c) = 0.

proof: let f(a) = f(b) = K; f is continuous on $[a, b] \implies$ there exists an absolute maximum point $c_1 \in [a, b]$ and an absolute minimum point $c_2 \in [a, b]$ (theorem 5.33)

- if $c_1 > K$, then $c_1 \in (a,b) \implies f'(c_1) = 0$ (theorem 6.11)
- if $c_2 < K$, then $c_2 \in (a,b) \implies f'(c_2) = 0$ (theorem 6.11)
- if $c_1 = c_2 = K$, then $K \le f(x) \le K$ for all $x \in [a, b] \implies f(x) = K$ for all $x \in [a, b] \implies f'(c) = 0$ for all $c \in (a, b)$

Theorem 6.15 Mean value theorem. Let the function $f: [a, b] \to \mathbf{R}$ be continuous and differentiable on (a, b), then there exists some $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

proof:

• define $g \colon [a,b] \to \mathbf{R}$ with $g(x) = f(x) - f(b) + \frac{f(b) - f(a)}{b-a}(b-x)$

• since g(a) = g(b) = 0, by theorem 6.14, there exists $c \in (a, b)$ such that

$$g'(c) = 0 = f'(c) - \frac{f(b) - f(a)}{b - a} \implies f(b) - f(a) = f'(c)(b - a)$$

Theorem 6.16 If the function $f: I \to \mathbf{R}$ is differentiable and f'(x) = 0 for all $x \in I$, then f is constant.

proof: let $a, b \in I$ with a < b, then f is continuous on [a, b] and differentiable on $(a, b) \implies \exists c \in (a, b)$ such that f(b) - f(a) = f'(c)(b - a) = 0 (since f'(x) = 0 for all $x \in I$) $\implies f(b) = f(a)$

Derivative

6-11

Theorem 6.17 Let $f: I \rightarrow \mathbf{R}$ be a differentiable function.

- The function f is increasing if and only if $f'(x) \ge 0$ for all $x \in I$.
- The function f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

proof: we prove the first statement

- suppose $f'(x) \ge 0$ for all $x \in I$, let $a, b \in I$ with a < b, then f is continuous on [a, b] and differentiable on $(a, b) \implies \exists c \in (a, b) \text{ s.t. } f(b) f(a) = f'(c)(b a)$ (theorem 6.15) and $f'(c) \ge 0 \implies f(b) f(a) \ge 0 \implies f(a) \le f(b)$
- suppose f is increasing, let $c \in I$, then we can find a sequence $(x_n)_{n=1}^{\infty}$ with either $x_n < c$ or $x_n > c$ for all $n \in \mathbb{N}$ such that $x_n \to c$

- if $x_n < c$ for all $n \in \mathbf{N} \implies f(x_n) \leq f(c)$ for all $n \in \mathbf{N}$, and hence

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \ge 0$$

- if $x_n > c$ for all $n \in \mathbf{N} \implies f(x_n) \ge f(c)$ for all $n \in \mathbf{N}$, and hence

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \ge 0$$

in either case, we have $f'(c) \ge 0$

Derivative

Definition 6.18 We say the function $f: I \to \mathbb{R}$ is *n*-times differentiable on $J \subseteq I$ if $f', f'', \ldots, f^{(n)}$ exist at every point in J, where $f^{(n)}$ denotes the *n*th derivative of f.

Theorem 6.19 Taylor. Suppose the function $f: [a, b] \to \mathbf{R}$ is continuous and has n continuous derivatives on [a, b] such that $f^{(n+1)}$ exists on (a, b). Given $x_0, x \in [a, b]$, there exists some $c \in (x_0, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0) (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.$$

We denote

$$P_n(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(x_0) (x - x_0)^k \text{ and } R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

as the nth order Taylor polynomial and the nth order remainder of f, respectively.

Derivative

proof: let $x, x_0 \in [a, b]$ and $x \neq x_0$ (if $x = x_0$ then any c satisfies the theorem) • let $M_{x,x_0} = \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}$, then we have

$$f(x) = P_n(x) + M_{x,x_0}(x - x_0)^{n+1}$$

- note that for all $0 \le k \le n$, we have $f^{(k)}(x_0) = P^{(k)}_n(x_0)$
- let $g(s) = f(s) P_n(s) M_{x,x_0}(s x_0)^{n+1}$, then we have

$$g(x_0) = f(x_0) - P_n(x_0) - M_{x,x_0}(x_0 - x_0)^{n+1} = 0$$

$$g'(x_0) = f'(x_0) - P'_n(x_0) - M_{x,x_0}(n+1)(x_0 - x_0)^n = 0$$

$$\vdots$$

$$g^{(n)}(x_0) = f^{(n)}(x_0) - P^{(n)}_n(x_0) - M_{x,x_0}(n+1)!(x_0 - x_0) = 0$$

• by theorem 6.15:

$$g(x_0) = g(x) = 0 \implies \exists x_1 \text{ between } x_0 \text{ and } x \text{ s.t. } g'(x_1) = 0$$

$$g'(x_0) = g'(x_1) = 0 \implies \exists x_2 \text{ between } x_0 \text{ and } x_1 \text{ s.t. } g''(x_2) = 0$$

$$\vdots$$

$$g^{(n-1)}(x_0) = g^{(n-1)}(x_{n-1}) = 0 \implies \exists x_n \text{ between } x_0 \text{ and } x_{n-1} \text{ s.t. } g^{(n)}(x_n) = 0$$

$$g^{(n)}(x_0) = g^{(n)}(x_n) = 0 \implies \exists c \text{ between } x_0 \text{ and } x_n \text{ s.t. } g^{(n+1)}(c) = 0$$

Derivative

note that

$$\frac{d^{n+1}}{ds^{n+1}}M_{x,x_0}(s-x_0)^{n+1} = M_{x,x_0}(n+1)! \quad \text{and} \quad P_n^{(n+1)}(c) = 0$$

• we have the (n + 1)-times derivative of g at c given by

$$0 = g^{(n+1)}(c) = f^{(n+1)}(c) - M_{x,x_0}(n+1)! \implies M_{x,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}$$

• hence, we have

$$f(x) = P_n(x) + M_{x,x_0}(x - x_0)^{n+1}$$
$$= P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Derivative

Theorem 6.20 Second derivative test. Suppose the function $f: (a, b) \to \mathbf{R}$ has two continuous derivatives. If $x_0 \in (a, b)$ such that $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a strict relative minimum at x_0 .

proof:

- it is easy to show that f'' is continuous and $f''(x_0) > 0 \implies$ there exists some $\delta > 0$ such that for all $c \in (x_0 \delta, x_0 + \delta)$, we have f''(c) > 0
- then for all $x \in (x_0 \delta, x_0 + \delta)$, by theorem 6.19, there exists some c_0 between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(c_0)(x - x_0)^2$$

• c_0 between x and $x_0 \implies c_0 \in (x_0 - \delta, x_0 + \delta) \implies f''(c) > 0$, and since $f'(x_0) = 0$, we have

$$f(x) - f(x_0) = \frac{1}{2}f''(c_0)(x - x_0)^2 > 0 \implies f(x) > f(x_0)$$

7. Riemann integral

- Riemann sum and some useful facts
- Riemann integral of continuous functions
- properties of Riemann integral
- fundamental theorem of calculus
- integration by parts
- change of variables

Riemann integral

Riemann sum

Definition 7.1 A partition $\underline{x} = \{x_0, x_1, \dots, x_n\}$ of [a, b] is a finite set such that

$$a = x_0 < x_1 < \dots < x_n = b.$$

The **norm** of \underline{x} , denoted $||\underline{x}||$, is a number defined as

$$\|\underline{x}\| = \max\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\}.$$

Definition 7.2 let \underline{x} be a partition of [a, b]. A tag of \underline{x} is a finite set $\underline{\xi} = \{\xi_1, \dots, \xi_n\}$ such that

$$a = x_0 \le \xi_1 \le x_1 \le \xi_2 \le x_2 \le \dots \le x_{n-1} \le \xi_n \le x_n = b.$$

The pair $(\underline{x}, \underline{\xi})$ is referred to as a **tagged partition**.

example: $(\underline{x}, \underline{\xi}) = (\{1, 3/2, 2, 3\}, \{5/4, 7/4, 5/2\})$ is a tagged partition with norm

 $||\underline{x}|| = \max\{3/2 - 1, 2 - 3/2, 3 - 2\} = 1$

Riemann integral

Definition 7.3 The **Riemann sum** of f corresponding to (\underline{x}, ξ) is the number

$$S_f(\underline{x},\underline{\xi}) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}).$$

Remark 7.4 For a continuous function f on [a, b] that is positive, the Riemann sum $S_f(\underline{x}, \underline{\xi})$ is an approximate area under the graph of f. As $||\underline{x}|| \to 0$, we should expect these approximate areas to converge to some number, which we **interpret** as the area under the graph of f on the interval [a, b].

Riemann integral

Some useful facts

Definition 7.5 We define the set $C([a, b]) = \{f : [a, b] \to \mathbf{R} \mid f \text{ is continuous}\}.$

Definition 7.6 Let $f \in C([a, b])$ and $\tau > 0$, we define the **modulus of continuity** of the function f as

$$w_f(\tau) = \sup\{|f(x) - f(y)| \mid |x - y| \le \tau\}.$$

Theorem 7.7 For all $f \in C([a, b])$, we have $\lim_{\tau \to 0} w_f(\tau) = 0$, *i.e.*, for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $\tau < \delta$, we have $w_f(\tau) < \epsilon$.

proof: let $\epsilon > 0$

- $f \in \mathcal{C}([a,b]) \implies f$ is uniformly continuous on $[a,b] \implies \exists \delta > 0$ such that for all $x, y \in [a,b]$ and $|x-y| < \delta$, we have $|f(x) f(y)| < \epsilon/2$
- let $\tau < \delta$, then for all $x, y \in [a, b]$ and $|x y| \le \tau$, we have $|x y| < \delta \implies |f(x) f(y)| < \epsilon/2$ for all $x, y \in [a, b]$ and $|x y| \le \tau \implies \epsilon/2$ is an upper bound of the set $\{|f(x) f(y)| \mid |x y| \le \tau\} \implies w_f(\tau) \le \epsilon/2 < \epsilon$

Theorem 7.8 Let $f \in \mathcal{C}([a, b])$, then $w_f(\tau)$ has the following properties:

- For all $x, y \in [a, b]$, we have $w_f(|x y|) \ge |f(x) f(y)|$.
- Monotonicity. If $\tau_1 \leq \tau_2$, then $w_f(\tau_1) \leq w_f(\tau_2)$.

Definition 7.9 Let $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ be tagged partitions of [a, b]. We say \underline{x}' is a refinement of \underline{x} if $\underline{x} \subseteq \underline{x}'$.

Theorem 7.10 Let $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ be tagged partitions of [a, b] such that \underline{x}' is a refinement of \underline{x} . If $f \in \overline{C}([a, b])$, then

$$|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| \le w_f(||\underline{x}||)(b-a).$$

proof: let $\underline{x} = \{x_0, \dots, x_n\}$, $\underline{\xi} = \{\xi_1, \dots, \xi_n\}$, $\underline{x}' = \{x'_0, \dots, x'_n\}$, $\underline{\xi}' = \{\xi'_1, \dots, \xi'_n\}$ • for $i = 1, \dots, n$, let $\underline{y}^{(i)} = \{x'_q, x'_{q+1}, \dots, x'_k\}$, $\underline{\zeta}^{(i)} = \{\xi'_{q+1}, \xi'_{q+2}, \dots, \xi'_k\}$ s.t. $x_{i-1} = x'_q < x'_{q+1} < \dots < x'_k = x_i$

Riemann integral

• then for all $i = 1, \ldots, n$, we have	ave	e ha∖	we	., <i>n</i> ,	· · · ·	=	ı	all	tor	then	•
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$$\begin{aligned} |f(\xi_{i})(x_{i} - x_{i-1}) - S_{f}(\underline{y}^{(i)}, \underline{\zeta}^{(i)})| \\ &= \left| f(\xi_{i}) \sum_{\ell=q+1}^{k} (x_{\ell}' - x_{\ell-1}') - \sum_{\ell=q+1}^{k} f(\xi_{\ell}')(x_{\ell}' - x_{\ell-1}') \right| \\ &= \left| \sum_{\ell=q+1}^{k} (f(\xi_{i}) - f(\xi_{\ell}'))(x_{\ell}' - x_{\ell-1}') \right| \leq \sum_{\ell=q+1}^{k} |f(\xi_{i}) - f(\xi_{\ell}')|(x_{\ell}' - x_{\ell-1}') \\ &\leq \sum_{\ell=q+1}^{k} w_{f}(x_{i} - x_{i-1})(x_{\ell}' - x_{\ell-1}') \leq \sum_{\ell=q+1}^{k} w_{f}(||\underline{x}||)(x_{\ell}' - x_{\ell-1}') \\ &= w_{f}(||\underline{x}||)(x_{i} - x_{i-1}) \end{aligned}$$
(7.1)

- the first inequality is by lemma 4.18
- the second inequality is from $\xi_i, \xi'_\ell \in [x_{i-1}, x_i]$
- the third inequality is by the second statement of theorem 7.8, and $\|\underline{x}\| \ge x_i x_{i-1}$

• put together, we have

$$|S_{f}(\underline{x},\underline{\xi}) - S_{f}(\underline{x}',\underline{\xi}')| = \left|\sum_{i=1}^{n} (f(\xi_{i})(x_{i} - x_{i-1}) - S_{f}(\underline{y}^{(i)},\underline{\zeta}^{(i)}))\right|$$

$$\leq \sum_{i=1}^{n} |f(\xi_{i})(x_{i} - x_{i-1}) - S_{f}(\underline{y}^{(i)},\underline{\zeta}^{(i)})| \leq \sum_{i=1}^{n} w_{f}(||\underline{x}||)(x_{i} - x_{i-1})$$

$$= w_{f}(||x||)(b - a),$$

where the last inequality is by plugging in (7.1)

Theorem 7.11 Let $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ be any two tagged partitions of [a, b] and $f \in C([a, b])$, then

$$|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| \le (w_f(||\underline{x}||) + w_f(||\underline{x}'||))(b-a).$$

proof: let $\underline{x}'' = \underline{x} \cup \underline{x}'$ and ξ'' be a tag of \underline{x}'' , then by theorem 7.10, we have

$$\begin{aligned} |S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')| &\leq |S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}'',\underline{\xi}'')| + |S_f(\underline{x}'',\underline{\xi}'') - S_f(\underline{x}',\underline{\xi}')| \\ &\leq (w_f(||\underline{x}||) + w_f(||\underline{x}'||))(b-a) \end{aligned}$$

Riemann integral

Riemann integral of continuous functions

Theorem 7.12 Let $f \in \mathcal{C}([a, b])$, then there exists a unique number denoted $\int_a^b f(x) dx$ with the following property: For all sequences of tagged partitions $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ such that $\lim_{r\to\infty} \|\underline{x}^{(r)}\| = 0$, we have

$$\lim_{r \to \infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \int_a^b f(x) \, dx.$$

proof: uniqueness follows immediately from uniqueness of limits of sequences of real numbers, we only need to show the existence

let ((<u>y</u>^(r), <u>ζ</u>^(r)))[∞]_{r=1} be a sequence of tagged partitions with lim_{r→∞} ||<u>y</u>^(r)|| = 0, we first show that (S_f(<u>y</u>^(r), <u>ζ</u>^(r)))[∞]_{r=1} is a Cauchy sequence; let ε > 0
by theorem 7.7, ∃δ > 0 such that for all τ < δ, w_f(τ) < ε/(2(b-a))
||<u>y</u>^(r)|| → 0 ⇒ ∃M ∈ N s.t. ∀r, s ≥ M, ||<u>y</u>^(r)|| < δ, ||<u>y</u>^(s)|| < δ ⇒ ∀r, s ≥ M, we have w_f(||<u>y</u>^(r)||) < ε/(2(b-a)), w_f(||<u>y</u>^(s)||) < ε/(2(b-a))

– hence, for all $r,s\geq M,$ by theorem 7.11, we have

$$|S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - S_f(\underline{y}^{(s)}, \underline{\zeta}^{(s)})|$$

$$\leq (w_f(||\underline{y}^{(r)}||) + w_f(||\underline{y}^{(s)}||))(b-a) < \left(\frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)}\right)(b-a) = \epsilon$$

let $L = \lim_{r \to \infty} S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)})$ (which exists by theorem 3.45)

• let
$$\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$$
 be any sequence of partitions with $\lim_{r\to\infty} \|\underline{x}^{(r)}\| = 0$, we
now show that $\lim_{r\to\infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = L$
- since $\|\underline{x}^{(r)}\| \to 0$, $\|\underline{y}^{(r)}\| \to 0$, by theorem 7.7, we have
 $\lim_{r\to\infty} (w_f(\|\underline{x}^{(r)}\|) + w_f(\|\underline{y}^{(r)}\|))(b-a) = 0$
- $S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) \to L \implies |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L| \to 0$
- by theorem 7.11, we have
 $0 \le |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - L| \le |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)})| + |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L|$
 $\le (w_f(\|\underline{x}^{(r)}\|) + w_f(\|\underline{y}^{(r)}\|))(b-a) + |S_f(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) - L|$
 $\implies \lim_{r\to\infty} |S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) - L| = 0$ (theorem 3.21)

Riemann integral

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Remark 7.13 Let $f \in \mathcal{C}([a, b])$. We sometimes write

$$\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx$$

By convention, we also define

$$\int_{a}^{a} f = 0 \quad \text{and} \quad \int_{b}^{a} f = -\int_{a}^{b} f.$$

Properties of Riemann integral

Theorem 7.14 Linearity. Let $f, g \in \mathcal{C}([a, b])$ and $\alpha \in \mathbf{R}$, then

$$\int_{a}^{b} (\alpha f + g) = \alpha \int_{a}^{b} f + \int_{a}^{b} g.$$

proof: let $((\underline{x}^{(r)}, \underline{\xi}^{(r)}))_{r=1}^{\infty}$ be a sequence of tagged partitions such that $||\underline{x}^{(r)}|| \to 0$, then we have

$$\int_{a}^{b} (\alpha f + g) = \lim_{r \to \infty} S_{\alpha f + g}(\underline{x}^{(r)}, \underline{\xi}^{(r)})$$
$$= \lim_{r \to \infty} (\alpha S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) + S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)}))$$
$$= \alpha \lim_{r \to \infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) + \lim_{r \to \infty} S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)})$$
$$= \alpha \int_{a}^{b} f + \int_{a}^{b} g$$

Riemann integral

Theorem 7.15 Additivity. Let $f \in C([a, b])$ and a < c < b, then we have

$$\int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f.$$

proof:

- let $\left((\underline{y}^{(r)}, \underline{\zeta}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions of [a, c] with $\|\underline{y}^{(r)}\| \to 0$
- let $((\underline{z}^{(r)}, \underline{\eta}^{(r)}))_{r=1}^{\infty}$ be a sequence of tagged partitions of [c, b] with $\|\underline{z}^{(r)}\| \to 0$
- then $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ with $\underline{x}^{(r)} = \underline{y}^{(r)} \cup \underline{z}^{(r)}$ and $\underline{\xi}^{(r)} = \underline{\zeta}^{(r)} \cup \underline{\eta}^{(r)}$ is a sequence of tagged partitions of [a, b]

•
$$\|\underline{y}^{(r)}\| \to 0$$
 and $\|\underline{z}^{(r)}\| \to 0 \implies \|\underline{x}^{(r)}\| \le \|\underline{y}^{(r)}\| + \|\underline{z}^{(r)}\| \to 0$

hence, we have

$$\int_{a}^{b} f = \lim_{r \to \infty} S_{f}(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \lim_{r \to \infty} (S_{f}(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) + S_{f}(\underline{z}^{(r)}, \underline{\eta}^{(r)}))$$
$$= \lim_{r \to \infty} S_{f}(\underline{y}^{(r)}, \underline{\zeta}^{(r)}) + \lim_{r \to \infty} S_{f}(\underline{z}^{(r)}, \underline{\eta}^{(r)}) = \int_{a}^{c} f + \int_{c}^{b} f$$

Riemann integral

Theorem 7.16 Let $f,g \in \mathcal{C}([a,b])$ and $f(x) \leq g(x)$ for all $x \in [a,b]$, then we have

$$\int_{a}^{b} f \le \int_{a}^{b} g.$$

proof: let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions with $\|\underline{x}^{(r)}\| \to 0$, then $S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) \le \sum_{i=1}^{n^{(r)}} g(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)})$ $\implies \lim_{r \to \infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \le \lim_{r \to \infty} S_g(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \implies \int_a^b f \le \int_a^b g$

Corollary 7.17 Let $f \in \mathcal{C}([a, b])$, then $\left|\int_{a}^{b} f\right| \leq \int_{a}^{b} |f|$.

proof:
$$\pm f(x) \le |f(x)| \implies \int_a^b \pm f = \pm \int_a^b f \le \int_a^b |f|$$
 (theorem 7.16)

Riemann integral

Theorem 7.18 Let $f \in \mathcal{C}([a,b])$, and

$$m_f = \inf\{f(x) \mid x \in [a, b]\}, \qquad M_f = \sup\{f(x) \mid x \in [a, b]\}.$$

Then, we have

$$m_f(b-a) \le \int_a^b f \le M_f(b-a).$$

proof: let $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ be a sequence of tagged partitions with $\|\underline{x}^{(r)}\| \to 0$, then

$$S_{f}(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_{i}^{(r)})(x_{i}^{(r)} - x_{i-1}^{(r)}) \ge \sum_{i=1}^{n^{(r)}} m_{f}(x_{i}^{(r)} - x_{i-1}^{(r)}) = m_{f}(b-a)$$

$$S_{f}(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_{i}^{(r)})(x_{i}^{(r)} - x_{i-1}^{(r)}) \le \sum_{i=1}^{n^{(r)}} M_{f}(x_{i}^{(r)} - x_{i-1}^{(r)}) = M_{f}(b-a)$$

$$\Rightarrow m_{f}(b-a) \le \lim_{r \to \infty} S_{f}(\underline{x}^{(r)}, \underline{\xi}^{(r)}) \le M_{f}(b-a)$$

Theorem 7.19 Fundamental theorem of calculus. Let $f \in C([a, b])$.

• If $F \colon [a,b] \to \mathbf{R}$ is differentiable and F' = f, then

$$\int_{a}^{b} f = F(b) - F(a).$$

• The function $G(x) = \int_a^x f$ is differentiable on [a, b] with

$$G(a) = 0, \qquad G'(x) = f(x).$$

proof:

• let $(\underline{x}^{(r)})_{r=1}^{\infty}$ be a sequence of partitions with $\|\underline{x}^{(r)}\| \to 0$, by theorem 6.15, there exist tags $\underline{\xi}^{(r)}$ with $\xi_i^{(r)} \in [x_{i-1}^{(r)}, x_i^{(r)}]$, $i = 1, \ldots, n^{(r)}$, such that

$$F(x_i^{(r)}) - F(x_{i-1}^{(r)}) = F'(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)})$$

Riemann integral

hence, for the sequence of tagged partitions $\left((\underline{x}^{(r)}, \underline{\xi}^{(r)})\right)_{r=1}^{\infty}$ we have

$$S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = \sum_{i=1}^{n^{(r)}} f(\xi_i^{(r)})(x_i^{(r)} - x_{i-1}^{(r)}) = \sum_{i=1}^{n^{(r)}} F(x_i^{(r)}) - F(x_{i-1}^{(r)}) = F(b) - F(a)$$

$$\implies \int_a^b f = \lim_{r \to \infty} S_f(\underline{x}^{(r)}, \underline{\xi}^{(r)}) = F(b) - F(a)$$

- we only need to show that G is differentiable and G' = f, *i.e.*, let $c \in [a, b]$, we need to prove that $\lim_{x\to c} \frac{G(x)-G(c)}{x-c} = \lim_{x\to c} \frac{\int_a^x f \int_a^c f}{x-c} = f(c)$; let $\epsilon > 0$ f continuous on $[a, b] \implies \exists \delta > 0$ such that for all $t \in [a, b]$ and $|t-c| < \delta$, we
 - have $|f(t) f(c)| < \epsilon/2$
 - suppose $x \in (c, c + \delta)$, then for all $t \in [c, x]$, we have $|f(t) f(c)| < \epsilon/2$, hence,

$$\begin{aligned} \left| \frac{\int_{a}^{x} f - \int_{a}^{c} f}{x - c} - f(c) \right| &= \left| \frac{\int_{c}^{x} f(t) dt}{x - c} - f(c) \right| \\ &= \left| \frac{1}{x - c} \left(\int_{c}^{x} f(t) dt - \int_{c}^{x} f(c) dt \right) \right| = \frac{1}{x - c} \left| \int_{c}^{x} (f(t) - f(c)) dt \right| \\ &\leq \frac{1}{x - c} \int_{c}^{x} |f(t) - f(c)| dt \leq \frac{1}{x - c} \int_{c}^{x} \frac{\epsilon}{2} dt = \frac{1}{x - c} \cdot \frac{\epsilon}{2} (x - c) = \frac{\epsilon}{2} < \epsilon \end{aligned}$$

(the first inequality is by corollary 7.17)

Riemann integral

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- suppose $x \in (c - \delta, c)$, using similar argument, we have $\left|\frac{\int_a^x f - \int_a^c f}{x - c} - f(c)\right| < \epsilon$ - put together, we conclude that for all $x \in [a, b]$ and $0 < |x - c| < \delta$, we have

$$\left| \frac{\int_{a}^{x} f - \int_{a}^{c} f}{x - c} - f(c) \right| < \epsilon$$
$$\implies \qquad \lim_{x \to c} \frac{G(x) - G(c)}{x - c} = \lim_{x \to c} \frac{\int_{a}^{x} f - \int_{a}^{c} f}{x - c} = f(c)$$

Riemann integral

Integration by parts

Theorem 7.20 Integration by parts. Suppose $f,g \in \mathcal{C}([a,b])$, $f',g' \in \mathcal{C}([a,b])$, then

$$\int_{a}^{b} f'g = (f(b)g(b) - f(a)g(a)) - \int_{a}^{b} fg'.$$

proof: let $F \in \mathcal{C}([a,b])$ with F(x) = f(x)g(x), by theorem 6.8, we have

$$F'(x) = f'(x)g(x) + f(x)g'(x),$$

and hence,

$$\int_{a}^{b} f'(x)g(x) \, dx + \int_{a}^{b} f(x)g'(x) \, dx = \int_{a}^{b} (f'(x)g(x) + f(x)g'(x)) \, dx$$
$$= \int_{a}^{b} F'(x) \, dx = F(b) - F(a) = f(b)g(b) - f(a)g(a)$$
$$\implies \int_{a}^{b} f'g = (f(b)g(b) - f(a)g(a)) - \int_{a}^{b} fg'$$

Riemann integral

Theorem 7.21 Change of variables. Let $f \in C([c,d])$ and $\varphi \colon [a,b] \to [c,d]$ be continuously differentiable with $\varphi(a) = c$ and $\varphi(b) = d$. Then, we have

$$\int_{c}^{d} f(u) \ du = \int_{a}^{b} f(\varphi(x))\varphi'(x) \ dx.$$

proof:

• let $F\colon [a,b]\to {\bf R}$ be a function with F'=f, then we have

$$\int_{c}^{d} f(u) \, du = F(d) - F(c)$$

• by theorem 6.9, we have

$$(F \circ \varphi)'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x),$$

and hence,

$$\int_a^b f(\varphi(x))\varphi'(x) \, dx = F(\varphi(b)) - F(\varphi(a)) = F(d) - F(c) = \int_c^d f(u) \, du$$

Riemann integral

8. Sequences of functions

- power series
- pointwise and uniform convergence
- interchange of limits
- Weierstrass M-test
- properties of power series

Sequences of functions

8-1

Power series

Definition 8.1 A power series about $x_0 \in \mathbf{R}$ is a series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Definition 8.2 Let $\sum_{m=0}^{\infty} a_m (x - x_0)^m$ be a power series, if the limit

$$R = \lim_{m \to \infty} |a_m|^{1/m}$$

exists, we define the radius of convergence ρ as

$$\rho = \begin{cases} 1/R & R > 0\\ \infty & R = 0 \end{cases}$$

Theorem 8.3 Let $\sum_{m=0}^{\infty} a_m (x - x_0)^m$ be a power series and $R = \lim_{m \to \infty} |a_m|^{1/m}$ exists. If R = 0, the series converges absolutely for all $x \in \mathbf{R}$. If R > 0, the series converges absolutely if $|x - x_0| < \rho$ and diverges if $|x - x_0| > \rho$.

proof: consider the root test (theorem 4.26), we have

$$L = \lim_{m \to \infty} |a_m (x - x_0)^m|^{1/m} = |x - x_0| \lim_{m \to \infty} |a_m|^{1/m} = R|x - x_0|$$

- suppose R = 0, then we have L = 0 < 1 for all $x \in \mathbf{R} \implies \sum_{m=0}^{\infty} a_m (x x_0)^m$ converges absolutely for all $x \in \mathbf{R}$
- suppose R > 0- if $|x - x_0| < \rho \implies L < R\rho = 1 \implies \sum_{m=0}^{\infty} a_m (x - x_0)^m$ converges absolutely - if $|x - x_0| > \rho \implies L > R\rho = 1 \implies \sum_{m=0}^{\infty} a_m (x - x_0)^m$ diverges

Sequences of functions

Remark 8.4 Let $\sum_{m=0}^{\infty} a_m (x - x_0)^m$ be a power series with radius of convergence ρ . Define $f: (x_0 - \rho, x_0 + \rho) \to \mathbf{R}$ such that

$$f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m,$$

then, the function f is the limit of a sequence of functions $(f_n)_{n=1}^{\infty}$, given by

$$f(x) = \lim_{n \to \infty} f_n(x), \quad f_n(x) = \sum_{m=0}^n a_m (x - x_0)^m$$

Example 8.5 Consider the geometric series $\sum_{m=0}^{\infty} x^m$ (which is a power series with $a_m = 1$, $x_0 = 0$), we have $f: (-1, 1) \to \mathbf{R}$ given by

$$f(x) = \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = \lim_{n \to \infty} f_n(x), \quad f_n(x) = \sum_{m=0}^n x^m.$$

Example 8.6 Exponential function. Consider the power series with $a_m = \frac{1}{m!}$, $x_0 = 0$, we have the exponential function $f(x): \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = \lim_{n \to \infty} f_n(x), \quad f_n(x) = \sum_{m=0}^n \frac{x^m}{m!}.$$

Remark 8.7 Based on remark 8.4, we may ask several questions.

- (1) Is the function f continuous?
- (2) If (1) is true, is f differentiable, and does $f' = \lim_{n \to \infty} f'_n$?
- (3) If (1) is true, does $\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$?

Sequences of functions

Pointwise convergence

Definition 8.8 Let $(f_n)_{n=1}^{\infty}$ with $f_n: S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be a sequence of functions, and let $f: S \to \mathbf{R}$ be a function. We say that $(f_n)_{n=1}^{\infty}$ converges pointwise (or just converges) to f if for all $x \in S$, we have $\lim_{n\to\infty} f_n(x) = f(x)$.

Example 8.9 Let $f_n(x) = x^n$ be defined on [0,1], then we have the sequence of functions $(f_n)_{n=1}^{\infty}$ converges pointwise to $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

proof:

• if
$$x \in [0, 1)$$
: $\lim_{n \to \infty} x^n = 0$

• if
$$x = 1$$
: $\lim_{n \to \infty} 1^n = 1$

Remark 8.10 A sequence of continuous functions may not converge pointwise to a continuous function.



Sequences of functions

Example 8.11 Let $f_n(x) \colon [0,1] \to \mathbf{R}$ be defined by

$$f_n(x) = \begin{cases} 4n^2x & x \in [0, \frac{1}{2n}] \\ 4n - 4n^2x & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in [\frac{1}{n}, 1], \end{cases}$$

then $(f_n)_{n=1}^{\infty}$ converges pointwise to f(x) = 0 ($x \in [0,1]$).

proof: if x = 0, we have $\lim_{n\to\infty} f_n(0) = 0$; if $x \in (0,1]$, then $\exists M \in [0,1]$ such that $\forall n \geq M$, $\frac{1}{n} < x$, and hence,



Sequences of functions

Uniform convergence

Definition 8.12 Let $(f_n)_{n=1}^{\infty}$ with $f_n \colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be a sequence of functions, and let $f: S \to \mathbf{R}$ be a function. We say that $(f_n)_{n=1}^{\infty}$ converges uniformly to f if for all $\epsilon>0$, there exists some $M\in {f N}$ such that for all $n\geq M$ and $x\in S$, we have $|f_n(x) - f(x)| < \epsilon.$

Theorem 8.13 Let $f: S \to \mathbf{R}$, $f_n: S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be functions. If the sequence of functions $(f_n)_{n=1}^{\infty}$ converges uniformly to f, then $(f_n)_{n=1}^{\infty}$ converges pointwise to f.

proof: let $c \in S$, $\epsilon > 0$

- $(f_n)_{n=1}^{\infty}$ converges uniformly to $f \implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, $|f_n(x) - f(x)| < \epsilon$
- hence, $\forall n \geq M$, $|f_n(c) f(c)| < \epsilon \implies (f_n)_{n=1}^{\infty}$ converges pointwise to f

Remark 8.14 Let $f: S \to \mathbf{R}$, $f_n: S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be functions. The sequence $(f_n)_{n=1}^{\infty}$ does not converge to f uniformly if there exists some $\epsilon > 0$ such that for all $M \in \mathbf{N}$, there exist some $n \ge M$ and some $x \in S$, so that $|f_n(x) - f(x)| \ge \epsilon$.

Theorem 8.15 Let $f_n(x) = x^n$, $n \in \mathbb{N}$, and let $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

- The sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to f on [0, b] for all 0 < b < 1.
- The sequence $(f_n)_{n=1}^{\infty}$ does not converges to f uniformly on [0,1].

proof:

• let $\epsilon > 0$, $b \in (0, 1)$, then $b^n \to 0 \implies \exists M \in \mathbf{N}$ such that $\forall n \ge M$, $b^n < \epsilon \implies \forall n \ge M$ and $x \in [0, b]$, we have

$$|f_n(x) - f(x)| = x^n \le b^n < \epsilon$$

• choose $\epsilon = 1/2$, then $\forall M \in \mathbf{N}$, choose n = M, $x = (1/2)^{1/M} < 1$, we have

$$|f_M(x) - f(x)| = x^M = 1/2 \ge \epsilon$$

Sequences of functions

Interchange of limits

Example 8.16 In general, limits cannot be interchanged. For example,

$$\lim_{n \to \infty} \lim_{k \to \infty} \frac{n/k}{n/k+1} = \lim_{n \to \infty} 0 = 0, \qquad \lim_{k \to \infty} \lim_{n \to \infty} \frac{n/k}{n/k+1} = \lim_{k \to \infty} 1 = 1.$$

Remark 8.17 Based on example 8.16, we may ask the following questions.

- If f_n: S → R with f_n continuous for all n ∈ N and (f_n)[∞]_{n=1} converges to f uniformly or pointwise, then is f continuous?
- If f_n: [a, b] → R with f_n differentiable for all n ∈ N, and (f_n)[∞]_{n=1} converges to f, (f'_n)[∞]_{n=1} converges to g uniformly or pointwise, then is f differentiable and does f' = g?
- If $f_n: [a,b] \to \mathbf{R}$, $n \in \mathbf{N}$, $f: [a,b] \to \mathbf{R}$, with f_n and f continuous, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly or pointwise, then does $\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$?

Remark 8.18 If convergence is only pointwise, the answer is no for all questions in remark 8.17.

- Let $f_n(x) = x^n$ on [0,1], $n \in \mathbb{N}$. Example 8.9 shows that $(f_n)_{n=1}^{\infty}$ converges pointwise to a noncontinuous function.
- Let $f_n(x) = \frac{x^{n+1}}{n+1}$ on [0,1], then $(f_n)_{n=1}^{\infty}$ converges to f(x) = 0 pointwise on $[0,1] \text{ and } (f'_n)_{n=1}^\infty$ converges pointwise to g given by $g(x) = \left\{ \begin{array}{cc} 0 & x \in [0,1) \\ 1 & x = 1 \end{array} \right.$, but $f'(1) = 0 \neq q(1) = 1$.
- Let $f_n: [0,1] \to \mathbf{R}$ be given by $f_n(x) = \begin{cases} 4n^2x & x \in \left[0, \frac{1}{2n}\right] \\ 4n 4n^2x & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & x \in \left[\frac{1}{n}, 1\right] \\ (f_n)_{n=1}^{\infty} \text{ converges to } f(x) = 0 \text{ pointwise on } [0,1] \text{ (example 8.11), but} \end{cases}$

$$\int_{0}^{1} f = 0 \neq \lim_{n \to \infty} \int_{0}^{1} f_{n} = \lim_{n \to \infty} (\frac{1}{2} \cdot \frac{1}{n} \cdot 2n) = 1.$$

Sequences of functions

Theorem 8.19 If $f_n: S \to \mathbf{R}$ is continuous for all $n \in \mathbf{N}$, $f: S \to \mathbf{R}$, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly, then f is continuous.

proof: let $c \in S$, $\epsilon > 0$

- f_n continuous on S, $c \in S \implies \exists \delta > 0$ such that for all $x \in S$ and $|x c| < \delta$, we have $|f_n(x) - f_n(c)| < \epsilon/3$
- $f_n \to f$ uniformly $\implies \exists M \in \mathbf{N}$ such that for all $n \ge M$ and $x \in S$, we have $|f(x) - f_n(x)| < \epsilon/3$
- hence, for all $x \in S$ and $|x c| < \delta$, we have

$$|f(x) - f(c)| = |f(x) - f_M(x) + f_M(x) - f_M(c) + f_M(c) - f(c)|$$

$$\leq |f(x) - f_M(x)| + |f_M(x) - f_M(c)| + |f_M(c) - f(c)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Theorem 8.20 If $f_n: [a,b] \to \mathbf{R}$ is continuous for all $n \in \mathbf{N}$, $f: [a,b] \to \mathbf{R}$, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly, then $\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$.

proof: let $\epsilon > 0$

- by theorem 8.19, we know that f is continuous on [a, b]
- $(f_n)_{n=1}^{\infty}$ converges uniformly to $f \implies \exists M \in \mathbf{N}$ such that for all $n \ge M$ and $x \in [a, b]$, we have $|f_n(x) f(x)| < \frac{\epsilon}{b-a}$
- hence, for all $n \ge M$, we have

$$\left|\int_{a}^{b} f_{n} - \int_{a}^{b} f\right| = \left|\int_{a}^{b} (f_{n} - f)\right| \le \int_{a}^{b} |f_{n} - f| < \int_{a}^{b} \frac{\epsilon}{b - a} = \epsilon$$

where the first inequality is by corollary 7.17

Remark 8.21 Notationally, theorem 8.20 says that

$$\int_{a}^{b} f = \int_{a}^{b} \lim_{n \to \infty} f_n = \lim_{n \to \infty} \int_{a}^{b} f_n.$$

Sequences of functions

Theorem 8.22 If $f_n: [a,b] \to \mathbf{R}$ is continuously differentiable, $f: [a,b] \to \mathbf{R}$, $g: [a,b] \to \mathbf{R}$, and

- $(f_n)_{n=1}^{\infty}$ converges to f pointwise,
- $(f'_n)_{n=1}^{\infty}$ converges to g uniformly,

then f is continuously differentiable and f' = g.

proof: let $x \in [a, b]$

- by theorem 8.19, we know that g is continuous on [a, b]
- by theorem 7.19, we have

$$\int_{a}^{x} f'_{n} = f_{n}(x) - f(a) \implies \lim_{n \to \infty} \int_{a}^{x} f'_{n} = \lim_{n \to \infty} f_{n}(x) - \lim_{n \to \infty} f_{n}(a)$$

- $f_n \to f$ pointwise $\implies \lim_{n \to \infty} f_n(x) \lim_{n \to \infty} f_n(a) = f(x) f(a)$
- $f'_n \to g$ uniformly $\implies \lim_{n \to \infty} \int_a^x f'_n = \int_a^x g$ (theorem 8.20)
- put together, we have

$$\int_{a}^{x} g = f(x) - f(a) \implies \left(\int_{a}^{x} g\right)' = g(x) = f'(x)$$

Sequences of functions

Theorem 8.23 Weierstrass M-test. Let $f_k \colon S \to \mathbf{R}$ for all $k \in \mathbf{N}$. Suppose there exists $M_k > 0$, $k \in \mathbf{N}$, such that

- (a) $|f_k(x)| \leq M_k$ for all $x \in S$,
- (b) $\sum_{k=1}^{\infty} M_k$ converges.

Then, we have the following conclusion.

- (1) The series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for all $x \in S$.
- (2) Let $f(x) = \sum_{k=1}^{\infty} f_k(x)$ for all $x \in S$, then the series $(\sum_{k=1}^{n} f_k)_{n=1}^{\infty}$ converges to f uniformly on S.

proof:

(1)
$$|f_k(x)| \le M_k$$
, $\sum_{k=1}^{\infty} M_k$ converges $\implies \sum_{k=1}^{\infty} |f_k(x)|$ converges (theorem 4.20) $\implies \sum_{k=1}^{\infty} f_k(x)$ converges absolutely

Sequences of functions

(2) let $\epsilon > 0$; $\sum_{k=1}^\infty M_k$ converges	$\implies \exists M \in \mathbf{N} \text{ s.t. } \forall n \geq M$,	we have
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$$\sum_{k=n+1}^{\infty} M_k = \left| \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k \right| < \epsilon$$

then, for all $n \geq M$ and $x \in S$, we have

$$\left|\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x)\right| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| \le \sum_{k=n+1}^{\infty} |f_k(x)| \le \sum_{k=n+1}^{\infty} M_k < \epsilon$$

Theorem 8.24 Let $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ be a power series with radius of convergence $\rho \in (0, \infty]$, then for all $r \in (0, \rho)$, the series $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ converges uniformly on $[x_0 - r, x_0 + r]$.

proof:

- note that we have $|x x_0| \le r$ for all $x \in [x_0 r, x_0 + r]$
- let $f_k = a_k(x x_0)^k$, choose $M_k = |a_k|r^k$, $k \in \mathbf{N}$, then $\forall x \in [x_0 r, x_0 + r]$,

$$|f_k(x)| = |a_k(x - x_0)^k| = |a_k||x - x_0|^k \le |a_k|r^k = M_k$$

• consider the root test (theorem 4.26) for $\sum_{k=0}^{\infty} M_k$, we have

$$L = \lim_{k \to \infty} M_k^{1/k} = \lim_{k \to \infty} \left(|a_k| r^k \right)^{1/k} = \lim_{k \to \infty} |a_k|^{1/k} r = \begin{cases} r/\rho & \rho < \infty \\ 0 & \rho = \infty \end{cases}$$

since $r \in (0, \rho)$, we have $L < 1 \implies \sum_{k=0}^{\infty} M_k$ converges absolutely

• put together, by theorem 8.23, we have $(\sum_{k=0}^{n} f_k)_{n=1}^{\infty} = \sum_{k=0}^{n} a_k (x - x_0)^k$ converges uniformly on $[x_0 - r, x_0 + r]$

Sequences of functions

Theorem 8.25 Let $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ be a power series with radius of convergence $\rho \in (0, \infty]$, then we have the following conclusion.

• For all $c \in (x_0 - \rho, x_0 + \rho)$, the function given by the series $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ is differentiable at c, and

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty}a_k(x-x_0)^k\right)\bigg|_{x=c} = \sum_{k=0}^{\infty}\frac{d}{dx}(a_k(x-x_0)^k)\bigg|_{x=c}$$

• For all a, b such that $x_0 - \rho < a < b < x_0 + \rho$,

$$\int_{a}^{b} \sum_{k=0}^{\infty} a_{k} (x - x_{0})^{k} dx = \sum_{k=0}^{\infty} \int_{a}^{b} a_{k} (x - x_{0})^{k} dx.$$

9. Metric spaces

- metric spaces
- Cauchy-Schwarz inequality
- open and closed sets
- closure and boundary
- sequences and convergence in metric spaces
- convergence properties of topology
- Cauchy sequences and completeness

Metric spaces

Metric spaces

Definition 9.1 Let A and B be sets. The **Cartesian product** is the set of tuples defined as

 $A \times B = \{(x, y) \mid x \in A, y \in B\}.$

examples:

- $\{a,b\} \times \{c,d\} = \{(a,c), (a,d), (b,c), (b,d)\}$
- the set $\mathbf{R}^2 = \mathbf{R} imes \mathbf{R}$ is the Cartesian plane
- the set $[0,1]^2 = [0,1] \times [0,1]$ is a subset of the Cartesian plane bounded by a square with vertices (0,0), (0,1), (1,0), and (1,1)

Remark 9.2 To denote an element in the set \mathbf{R}^n , we write $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, or simply $x \in \mathbf{R}^n$, where the subscripts $i = 1, \ldots, n$ denote the *i*th entry of the tuple (x_1, \ldots, x_n) that describes x.

We also simply write $0 \in \mathbf{R}^n$ to mean the point $(0, 0, \dots, 0) \in \mathbf{R}^n$.

Definition 9.3 Let X be a set, and let $d: X \times X \to \mathbf{R}$ be a function such that for all $x, y, z \in X$, we have

- $d(x,y) \ge 0$, (nonnegativity)
- d(x,y) = 0 if and only if x = y,
- d(x,y) = d(y,x), and (symmetry)
- $d(x,z) \le d(x,y) + d(y,z)$. (triangle inequality)

Then the pair (X, d) is called a **metric space**. The function d is called the **metric** or the **distance function**. Sometimes we just write X as the metric space if the metric is clear from context.

Example 9.4 The real numbers **R** is a metric space with the metric d(x, y) = |x - y|.

proof:

- the first three properties follows immediately from the properties of the absolute value (theorem 2.25)
- to show the triangle inequality, let $x, y, z \in \mathbf{R}$, then we have

$$|d(x,z) = |x-z| = |x-y+y-z| \le |x-y| + |y-z| = d(x,y) + d(x,z)$$

Metric spaces

Definition 9.5 Let (X, d) be a metric space. A set $S \subseteq X$ is said to be **bounded** if there exists a point $p \in X$ and some number $B \in \mathbf{R}$ such that

$$d(p,x) \leq B$$
 for all $x \in S$.

We say (X, d) is bounded if X is a bounded set.

Theorem 9.6 Cauchy-Schwarz inequality. Suppose $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, then

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \le \left(\sum_{i=1}^n x_i^2\right) \left(\sum_{i=1}^n y_i^2\right).$$

proof:

$$0 \le \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j - x_j y_i)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i^2 y_j^2 - 2x_i y_j x_j y_i + x_j^2 y_i^2)$$

= $\left(\sum_{i=1}^{n} x_i^2\right) \left(\sum_{j=1}^{n} y_j^2\right) + \left(\sum_{i=1}^{n} y_i^2\right) \left(\sum_{j=1}^{n} x_j^2\right) - 2\left(\sum_{i=1}^{n} x_i y_i\right) \left(\sum_{j=1}^{n} x_j y_j\right)$
 $\implies \left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right) \left(\sum_{i=1}^{n} y_i^2\right)$

Metric spaces

9-5

Theorem 9.7 The function $f \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ given by

$$f(x,y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

is a metric for \mathbf{R}^n .

proof: we show that f satisfies the triangle inequality, by theorem 9.6, we have

$$(f(x,z))^{2} = \sum_{i=1}^{n} (x_{i} - z_{i})^{2} = \sum_{i=1}^{n} (x_{i} - y_{i} + y_{i} - z_{i})^{2}$$

$$= \sum_{i=1}^{n} (x_{i} - y_{i})^{2} + 2\sum_{i=1}^{n} (x_{i} - y_{i})(y_{i} - z_{i}) + \sum_{i=1}^{n} (y_{i} - z_{i})^{2}$$

$$\leq \sum_{i=1}^{n} (x_{i} - y_{i})^{2} + 2\sqrt{\sum_{i=1}^{n} (x_{i} - y_{i})^{2} \sum_{i=1}^{n} (y_{i} - z_{i})^{2}} + \sum_{i=1}^{n} (y_{i} - z_{i})^{2}$$

$$= \left(\sqrt{\sum_{i=1}^{n} (x_{i} - y_{i})^{2}} + \sqrt{\sum_{i=1}^{n} (y_{i} - z_{i})^{2}}\right)^{2} = (f(x, y) + f(y, z))^{2}$$

Metric spaces

Definition 9.8 The *n*-dimensional Euclidean space is the metric space (\mathbf{R}^n, d) with the metric *d* defined by

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}.$$
 (9.1)

Remark 9.9 For n = 1, the *n*-dimensional Euclidean space reduces to the real numbers and the metric given by (9.1) agrees with the standard metric for the set of real numbers d(x, y) = |x - y| in example 9.4.

Metric spaces

Open and closed sets

Definition 9.10 Let (X, d) be a metric space, $x \in X$, and $\delta > 0$. Define the **open ball** and **closed ball**, of radius δ around x as

$$B(x,\delta) = \{y \in X \mid d(x,y) < \delta\} \text{ and } C(x,\delta) = \{y \in X \mid d(x,y) \le \delta\},\$$

respectively.

Example 9.11 Consider the metric space \mathbf{R} , for $x \in \mathbf{R}$ and $\delta > 0$, we have

$$B(x,\delta) = (x - \delta, x + \delta)$$
 and $C(x,\delta) = [x - \delta, x + \delta].$

Example 9.12 Consider the metric space \mathbf{R}^2 , for $x \in \mathbf{R}^2$ and $\delta > 0$, we have

$$B(x,\delta) = \{ y \in \mathbf{R}^2 \mid (x_1 - y_1)^2 + (x_2 - y_2)^2 < \delta^2 \}.$$

~

Definition 9.13 Let (X, d) be a metric space. A subset $V \subseteq X$ is **open** if for all $x \in V$, there exists some $\delta > 0$ such that $B(x, \delta) \subseteq V$. A subset $E \subseteq X$ is **closed** if the complement $E^c = X \setminus E$ is open.

examples:

- $(0,\infty) \subseteq \mathbf{R}$ is open; $[0,\infty) \subseteq \mathbf{R}$ is closed
- $[0,1) \subseteq \mathbf{R}$ is neither open nor closed
- the singleton $\{x\}$ with $x \in X$ is closed

Theorem 9.14 Let (X, d) be a metric space.

- (1) The sets \emptyset and X are open.
- (2) If V_1, \ldots, V_k are subsets of X, then $\bigcap_{i=1}^k V_i$ is open, *i.e.*, a *finite* intersection of open sets is open.
- (3) Let {V_i ⊆ X | i ∈ I} be a collection of open subsets of X, where I is an arbitrary index set, then U_{i∈I} V_i is open, *i.e.*, a union of open sets is open.

proof:

• the sets \emptyset and X are obviously open

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- let $x \in \bigcap_{i=1}^{k} V_i$, then $x \in V_1, \dots, V_k$ - V_1, \dots, V_k are open $\implies \exists \delta_1, \dots, \delta_k > 0$ s.t. $B(x, \delta_1) \subseteq V_1, \dots, B(x, \delta_k) \subseteq V_k$
 - choose $\delta = \min\{\delta_1, \ldots, \delta_k\}$, then $B(x, \delta) \subseteq V_1, \ldots, V_k \implies B(x, \delta) \subseteq \bigcap_{i=1}^k V_i$
- let $x \in \bigcup_{i \in I} V_i$, then $\exists V_k \in \{V_i \mid i \in I\}$ such that $x \in V_k$ - V_k is open $\implies \exists \delta > 0$ such that $B(x, \delta) \subseteq V_k \subseteq \bigcup_{i \in I} V_i$

Theorem 9.15 Let (X, d) be a metric space.

(1) The sets \emptyset and X are closed.

- (3) Let {V_i ⊆ X | i ∈ I} be a collection of closed subsets of X, where I is an arbitrary index set, then ∩_{i∈I} V_i is closed, *i.e.*, an intersection of closed sets is closed.
- (2) If V_1, \ldots, V_k are subsets of X, then $\bigcup_{i=1}^k V_i$ is closed, *i.e.*, a *finite* union of closed sets is closed.

Remark 9.16 Note that in theorem 9.14, the statement (2) is not true for an arbitrary intersection. For example, $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$, which is not open in **R**.

Similarly, in theorem 9.15, the statement (3) is not true for an arbitrary intersection. For example, $\bigcup_{n=1}^{\infty} [1/n, \infty) = (0, \infty)$, which is not closed in **R**.

Theorem 9.17 Let (X, d) be a metric space, $x \in X$, and $\delta > 0$. Then $B(x, \delta)$ is open and $C(x, \delta)$ is closed.

proof: we show that $B(x, \delta)$ is open; let $z \in B(x, \delta)$, then $d(x, z) < \delta$

- choose $\epsilon = \delta d(x, z)$, let $B(z, \epsilon) = \{y \in X \mid d(y, z) < \epsilon\}$ be an open ball
- let $y \in B(z,\epsilon)$, we have $d(y,z) < \epsilon$, and hence

$$d(x,y) \le d(x,z) + d(z,y) < d(x,z) + \epsilon = d(x,z) + \delta - d(x,z) = \delta$$

 $\implies y \in B(x,\delta) \implies B(z,\epsilon) \subseteq B(x,\delta)$

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Closure and boundary

Definition 9.18 Let (X, d) be a metric space and $A \subseteq X$. The closure of A is the set

 $\operatorname{cl} A = \bigcap \{ E \subseteq X \mid E \text{ is closed and } A \subseteq E \},$

i.e., $\mathbf{cl} A$ is the intersection of all closed sets that contain A.

Definition 9.19 Let (X, d) be a metric space and $A \subseteq X$. The **interior** of A is the set

int
$$A = \{x \in A \mid B(x, \delta) \subseteq A \text{ for some } \delta > 0\}.$$

The **boundary** of A is the set

$$\mathbf{bd}\,A = \mathbf{cl}\,A \setminus \mathbf{int}\,A.$$

example: consider A = (0, 1] and $X = \mathbf{R}$, then we have $\mathbf{cl} A = [0, 1]$, $\mathbf{int} A = (0, 1)$, and $\mathbf{bd} A = \{0, 1\}$

Remark 9.20 Notationally, in some textbooks, the closure, interior, and boundary of some set A are denoted as

$$\overline{A} = \mathbf{cl} A, \quad A^{\circ} = \mathbf{int} A, \quad \mathbf{and} \quad \partial A = \mathbf{bd} A,$$

respectively.

Theorem 9.21 Let (X, d) be a metric space and $A \subseteq X$.

- The closure $\mathbf{cl} A$ is closed and $A \subseteq \mathbf{cl} A$.
- If A is closed, then $\mathbf{cl} A = A$.

proof: let $\mathbf{cl} A = \bigcap \{ E \subseteq X \mid E \text{ is closed and } A \subseteq E \}$

- the first statement follows directly from the definition of closure and theorem 9.15
- if A is closed, then $A \in \{E \subseteq X \mid E \text{ is closed and } A \subseteq E\} \implies \mathbf{cl} A \subseteq A \implies A = \mathbf{cl} A$

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Theorem 9.22 Let (X, d) be a metric space and $A \subseteq X$, then $x \in \mathbf{cl} A$ if and only if for all $\delta > 0$, we have $B(x, \delta) \cap A \neq \emptyset$.

proof: we show the following claim: $x \notin \mathbf{cl} A$ if and only if there exists some $\delta > 0$ such that $B(x, \delta) \cap A = \emptyset$

- suppose $x \notin \mathbf{cl} A$, then $x \in (\mathbf{cl} A)^c$ - $\mathbf{cl} A$ is closed $\implies (\mathbf{cl} A)^c$ is open $\implies \exists \delta > 0$ s.t. $B(x, \delta) \subseteq (\mathbf{cl} A)^c \subseteq A^c \implies B(x, \delta) \cap A = \emptyset$
- suppose $\exists \delta > 0$ such that $B(x, \delta) \cap A = \emptyset$, let $x \in X$
 - $B(x,\delta)$ is open $\implies (B(x,\delta))^c$ is closed
 - $\ B(x,\delta) \cap A = \emptyset \implies A \subseteq (B(x,\delta))^c \implies \mathbf{cl} A \subseteq (B(x,\delta))^c$

$$-x \in B(x,\delta) \implies x \notin (B(x,c))^{\prime}$$

- put together, we have $x \notin \mathbf{cl} A$

Theorem 9.23 Let (X, d) be a metric space and $A \subseteq X$, then int A is open and bd A is closed.

proof:

- let $x \in \operatorname{int} A$
 - $\label{eq:alpha} \ x \in \operatorname{int} A \implies \exists \delta > 0 \text{ such that } B(x, \delta) \subseteq A$
 - let $z \in B(x, \delta)$; $B(x, \delta)$ open $\implies \exists \epsilon > 0$ such that $B(z, \epsilon) \subseteq B(x, \delta) \subseteq A \implies z \in \operatorname{int} A \implies B(x, \delta) \subseteq \operatorname{int} A \implies \operatorname{int} A$ is open
- int A open \implies (int A)^c closed \implies bd $A = cl A \setminus int A = cl A \cap (int A)^c$ is closed (theorem 9.15)

Theorem 9.24 Let (X, d) be a metric space and $A \subseteq X$, then $x \in \mathbf{bd} A$ if and only if for all $\delta > 0$, we have the sets $B(x, \delta) \cap A$ and $B(x, \delta) \cap A^c$ are both nonempty.

proof:

- suppose $x \in \mathbf{bd} A$, let $\delta > 0$
 - $\ x \in \mathbf{bd} \, A \implies x \in \mathbf{cl} \, A \text{, and hence, by theorem 9.22, we have } B(x, \delta) \cap A \neq \emptyset$
 - assume $B(x,\delta) \cap A^c = \emptyset$, then we have $B(x,\delta) \subseteq A \implies x \in int A$, which is a contradiction

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- suppose $B(x, \delta) \cap A \neq \emptyset$ and $B(x, \delta) \cap A^c \neq \emptyset$ for all $\delta > 0$, assume $x \notin \mathbf{bd} A$ - $x \notin \mathbf{bd} A \implies x \notin \mathbf{cl} A$ or $x \in \mathbf{int} A$
 - if $x \notin \mathbf{cl} A \implies \exists \delta_0 > 0$ such that $B(x, \delta_0) \cap A = \emptyset$, which is a contradiction
 - if $x \in \operatorname{int} A \implies \exists \delta_0 > 0$ such that $B(x, \delta_0) \subseteq A \implies B(x, \delta_0) \cap A^c = \emptyset$, which is a contradiction

Theorem 9.25 Let (X, d) be a metric space and $A \subseteq X$, then $\mathbf{bd} A = \mathbf{cl} A \cap \mathbf{cl}(A^c)$.

proof: let $x \in \mathbf{bd} A$, $\delta > 0$

- by theorem 9.24, we have $B(x, \delta) \cap A$ and $B(x, \delta) \cap A^c$ nonempty
- by theorem 9.22, $B(x, \delta) \cap A \neq \emptyset \implies x \in \mathbf{cl} A \text{ and } B(x, \delta) \cap A^c \neq \emptyset \implies x \in \mathbf{cl} A^c$
- hence, we have $\mathbf{bd} A = \mathbf{cl} A \cap \mathbf{cl}(A^c)$

Definition 9.26 A sequence in a metric space (X, d) is a function $x: \mathbb{N} \to X$. To denote a sequence we write $(x_n)_{n=1}^{\infty}$, where x_n is the *n*th element in the sequence.

A sequence $(x_n)_{n=1}^{\infty}$ is **bounded** if there exists a point $p \in X$ and $B \in \mathbf{R}$ such that $d(p, x_n) \leq B$ for all $n \in \mathbf{N}$.

Let $(n_i)_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers, then the sequence $(x_{n_i})_{i=1}^{\infty}$ is called a **subsequence** of $(x_n)_{n=1}^{\infty}$.

Definition 9.27 A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) is said to **converge** to a point $p \in X$ if for all $\epsilon > 0$, there exists some $M \in \mathbb{N}$ such that for all $n \ge M$, we have $d(x_n, p) < \epsilon$.

The point p is called a **limit** of $(x_n)_{n=1}^{\infty}$. If the limit p is unique, we write

$$\lim_{n \to \infty} x_n = p.$$

A sequence that converges is said to be **convergent**, and otherwise is **divergent**.

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Theorem 9.28 A convergent sequence in a metric space has a unique limit.

proof: let $x, y \in X$ such that $x_n \to x$ and $x_n \to y$; let $\epsilon > 0$

- $x_n \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall n \ge M_1$, $d(x_n, x) < \epsilon/2$
- $x_n o y \implies \exists M_2 \in \mathbf{N}$ such that $\forall n \geq M_2$, $d(x_n,y) < \epsilon/2$
- hence, for all $n \ge M$, we have

$$d(x,y) \le d(x_n,x) + d(x_n,y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\implies d(x,y) = 0 \implies x = y$$

Theorem 9.29 A convergent sequence in a metric space is bounded.

proof: suppose $x_n \to p \in X$

- let $\epsilon > 0$, $x_n \to p \implies \exists M \in \mathbf{N}$ such that $\forall n \ge M$, $d(x_n, p) < \epsilon$
- choose $B = \max\{d(x_1, p), \ldots, d(x_M, p), \epsilon\}$, then for all $n \in \mathbb{N}$, $d(x_n, p) \leq B$

Theorem 9.30 A sequence $(x_n)_{n=1}^{\infty}$ in a metric space (X, d) converges to $p \in X$ if and only if there exists a sequence $(a_n)_{n=1}^{\infty}$ of real numbers such that for all $n \in \mathbf{N}$, we have

$$d(x_n, p) \le a_n$$
 and $\lim_{n \to \infty} a_n = 0.$

proof:

• suppose
$$x_n \to p$$

 $-x_n \to p \implies \forall \epsilon > 0, \exists M \in \mathbb{N} \text{ s.t. } \forall n \ge M, \ d(x_n, p) < \epsilon \implies d(x_n, p) \to 0$
 $- \text{ choose } a_n = d(x_n, p) \text{ for all } n \in \mathbb{N}, \text{ then we have } d(x_n, p) \le a_n \text{ and } a_n \to 0$

• suppose
$$a_n \to 0$$
 with $a_n \in \mathbf{R}$ and $d(x_n, p) \le a_n$, let $\epsilon > 0$
- $0 \le d(x_n, p) \le a_n$, $a_n \to 0 \implies d(x_n, p) \to 0$ (theorem 3.21)

$$- d(x_n, p) \to 0 \implies \exists M \in \mathbf{N} \text{ such that } \forall n \ge M, \ d(x_n, p) < \epsilon \implies x_n \to p$$

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Theorem 9.31 Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (X, d). If $(x_n)_{n=1}^{\infty}$ converges to $p \in X$, then all subsequences of $(x_n)_{n=1}^{\infty}$ converges to p.

proof: let $\epsilon > 0$

- let $x_n \to p$, then $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $d(x_n, p) < \epsilon$
- let $(x_{n_i})_{i=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$, then we have $n_i \ge i$
- $\bullet \ \text{hence, for all} \ i \geq M \text{, we have} \ n_i \geq M \implies \forall i \geq M \text{, } d(x_{n_i},p) < \epsilon$

Theorem 9.32 Let $(x_n)_{n=1}^{\infty}$ be a sequence in \mathbb{R}^k , where $x_n \in \mathbb{R}^k$ for all $n \in \mathbb{N}$. Then $(x_n)_{n=1}^{\infty}$ converges if and only if $(x_{n,i})_{n=1}^{\infty}$ converges for all $i = 1, \ldots, k$, *i.e.*,

$$\lim_{n \to \infty} x_n = \left(\lim_{n \to \infty} x_{n,1}, \dots, \lim_{n \to \infty} x_{n,k} \right).$$

proof:

- suppose $x_n \to p \in \mathbf{R}^k$, let $\epsilon > 0$ - $x_n \to p \implies \exists M \in \mathbf{N}$ such that $\forall n \ge M$, $d(x_n, p) < \epsilon$
 - hence, $\forall n \geq M$, we have

$$(d(x_n, p))^2 = \sum_{i=1}^k (x_{n,i} - p_i)^2 < \epsilon^2 \implies (x_{n,i} - p_i)^2 < \epsilon^2, \quad i = 1, \dots, k$$
$$\implies |x_{n,i} - p_i| < \epsilon \text{ for all } i = 1, \dots, k \implies x_{n,i} \rightarrow p_i \text{ for all } i = 1, \dots, k$$

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suppose $x_{n,i} o p_i$ for all $i = 1, \dots, k$, let $\epsilon > 0$, $p = (p_1, \dots, p_k)$
$-x_{n,i} o p_i, \ i=1,\ldots,k \implies \exists M_1,\ldots,M_k \in \mathbf{N}$ such that $orall n \geq M_i$, we have
$ x_{n,i}-p_i <\epsilon/\sqrt{k},\ i=1,\ldots,k$

- choose
$$M = \max\{M_1, \ldots, M_k\}$$
, then $\forall n \ge M$, we have

$$d(x_n, p) = \sqrt{\sum_{i=1}^k (x_{n,i} - p_i)^2} < \sqrt{\sum_{i=1}^k \left(\frac{\epsilon}{\sqrt{k}}\right)^2} = \sqrt{\sum_{i=1}^k \frac{\epsilon^2}{k}} = \sqrt{\epsilon^2} = \epsilon$$
$$\implies x_n \to p$$

Theorem 9.33 Let (X, d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a sequence in X, then $(x_n)_{n=1}^{\infty}$ converges to $p \in X$ if and only if for all open sets $U \subseteq X$ with $p \in U$, there exists some $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n \in U$.

proof:

- suppose $x_n \to p$, let $U \subseteq X$ be open and $p \in U$
 - U is an open set contains $p \implies \exists \delta > 0$ such that $B(p, \delta) \subseteq U$
 - $\begin{array}{l} \ x_n \rightarrow p \implies \exists M \in \mathbf{N} \text{ s.t. } \forall n \geq M \text{, } d(x_n,p) < \delta \implies \forall n \geq M \text{, } x_n \in B(p,\delta) \implies \forall n \geq M \text{, } x_n \in U \end{array}$
- suppose for all open sets $U \subseteq X$ with $p \in U$, there exists some $M \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq M$; let $\epsilon > 0$
 - choose $U = B(p, \epsilon)$, then $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $x_n \in B(p, \epsilon)$
 - hence, $\forall n \geq M$, $d(x_n, p) < \epsilon \implies x_n \rightarrow p$

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Theorem 9.34 Let (X, d) be a metric space, $E \subseteq X$ be a closed set, and $(x_n)_{n=1}^{\infty}$ be a sequence in E that converges to some $p \in X$, then we have $p \in E$.

proof: assume $(x_n)_{n=1}^{\infty}$ in E converges to p but $p \notin E$

- $p \notin E \implies p \in E^c$
- E is closed $\implies E^c$ is open, then by theorem 9.33, $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $x_n \in E^c \implies \forall n \geq M$, $x_n \notin E$, which is a contradiction

Theorem 9.35 Let (X,d) be a metric space and $A \subseteq X$, then $p \in \mathbf{cl} A$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ of elements in A such that $\lim_{n\to\infty} x_n = p$.

proof:

suppose p ∈ cl A, then by theorem 9.22, we have B(p, δ) ∩ A ≠ Ø for all δ > 0
- choose (x_n)[∞]_{n=1} such that x_n ∈ A and d(x_n, p) < ¹/_n for all n ∈ N
- 0 ≤ d(x_n, p) < ¹/_n and ¹/_n → 0 ⇒ d(x_n, p) → 0 ⇒ x_n → p (theorem 9.30)

• suppose
$$(x_n)_{n=1}^{\infty}$$
 in A and $x_n \to p$, let $\delta > 0$
- $x_n \to p \implies \exists M \in \mathbf{N} \text{ s.t. } \forall n \ge M, \ d(x_n, p) < \delta \implies \forall n \ge M, \ x_n \in B(p, \delta)$

- then, since $x_n \in A$, we have $B(p, \delta) \cap A \neq \emptyset \implies p \in \operatorname{cl} A$ (theorem 9.22)
Definition 9.36 Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is **Cauchy** if for all $\epsilon > 0$, there exists some $M \in \mathbb{N}$ such that for all $n, k \ge M$, we have $d(x_n, x_k) < \epsilon$.

Theorem 9.37 A convergent sequence in a metric space is Cauchy.

proof: let $x_n \to p$, $\epsilon > 0$, then $\exists M \in \mathbf{N}$ such that $\forall n, k \ge M$, $d(x_n, p) < \epsilon/2$ and $d(x_k, p) < \epsilon/2$, and hence $\forall n, k \ge M$, we have

$$d(x_n, x_k) \le d(x_n, p) + d(x_k, p) < \epsilon/2 + \epsilon/2 = \epsilon$$

Definition 9.38 We say a metric space (X, d) is **complete** or **Cauchy-complete** if all Cauchy sequences in X converges to some point in X.

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Theorem 9.39 The Euclidean space \mathbf{R}^k is a complete metric space.

- **proof:** let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence with $x_n \in \mathbf{R}^k$ for all $n \in \mathbf{N}$; let $\epsilon > 0$
 - $(x_n)_{n=1}^{\infty}$ is Cauchy $\implies \exists M \in \mathbf{N}$ such that $\forall m, n \geq M$, $d(x_m x_n) < \epsilon$
 - hence, for all $m, n \ge M$, we have

$$(d(x_m, x_n))^2 = \sum_{i=1}^k (x_{m,i} - x_{n,i})^2 < \epsilon^2 \implies |x_{m,i} - x_{n,i}| < \epsilon, \quad i = 1, \dots, k$$

 \implies the sequence of real numbers $(x_{n,i})_{n=1}^{\infty}$ is Cauchy for all $i=1,\ldots,k$

- by theorem 3.45, we conclude that $(x_{n,i})_{n=1}^{\infty}$ converges for all $i = 1, \ldots, k$
- then, by theorem 9.32, we conclude that the sequence $(x_n)_{n=1}^\infty$ converges

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