8. Sequences of functions

- power series
- pointwise and uniform convergence
- interchange of limits
- Weierstrass M-test
- properties of power series

Power series

Definition 8.1 A **power series** about $x_0 \in \mathbf{R}$ is a series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Definition 8.2 Let $\sum_{m=0}^{\infty} a_m (x - x_0)^m$ be a power series, if the limit

$$R = \lim_{m \to \infty} |a_m|^{1/m}$$

exists, we define the **radius of convergence** ρ as

$$\rho = \begin{cases} 1/R & R > 0 \\ \infty & R = 0. \end{cases}$$

Theorem 8.3 Let $\sum_{m=0}^{\infty} a_m(x-x_0)^m$ be a power series and $R = \lim_{m\to\infty} |a_m|^{1/m}$ exists. If $R = 0$, the series converges absolutely for all $x \in \mathbf{R}$. If $R > 0$, the series converges absolutely if $|x - x_0| < \rho$ and diverges if $|x - x_0| > \rho$.

proof: consider the root test (theorem 4.26), we have

$$L = \lim_{m\to\infty} |a_m(x-x_0)^m|^{1/m} = |x-x_0| \lim_{m\to\infty} |a_m|^{1/m} = R|x-x_0|$$

- suppose $R = 0$, then we have $L = 0 < 1$ for all $x \in \mathbf{R} \implies \sum_{m=0}^{\infty} a_m(x-x_0)^m$ converges absolutely for all $x \in \mathbf{R}$
- suppose $R > 0$
  - if $|x-x_0| < \rho \implies L < R\rho = 1 \implies \sum_{m=0}^{\infty} a_m(x-x_0)^m$ converges absolutely
  - if $|x-x_0| > \rho \implies L > R\rho = 1 \implies \sum_{m=0}^{\infty} a_m(x-x_0)^m$ diverges

Remark 8.4 Let $\sum_{m=0}^{\infty} a_m(x-x_0)^m$ be a power series with radius of convergence ρ. Define $f\colon (x_0-\rho, x_0+\rho) \to \mathbf{R}$ such that

$$f(x) = \sum_{m=0}^{\infty} a_m(x-x_0)^m,$$

then, the function f is the limit of a sequence of functions $(f_n)_{n=1}^{\infty}$, given by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad f_n(x) = \sum_{m=0}^{n} a_m(x-x_0)^m.$$

Example 8.5 Consider the geometric series $\sum_{m=0}^{\infty} x^m$ (which is a power series with $a_m = 1$, $x_0 = 0$), we have $f\colon (-1,1) \to \mathbf{R}$ given by

$$f(x) = \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = \lim_{n\to\infty} f_n(x), \quad f_n(x) = \sum_{m=0}^{n} x^m.$$

Example 8.6 *Exponential function.* Consider the power series with $a_m = \frac{1}{m!}$, $x_0 = 0$, we have the exponential function $f(x)\colon \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = \lim_{n\to\infty} f_n(x), \quad f_n(x) = \sum_{m=0}^{n} \frac{x^m}{m!}.$$

Remark 8.7 Based on remark 8.4, we may ask several questions.

(1) Is the function f continuous?

(2) If (1) is true, is f differentiable, and does $f' = \lim_{n\to\infty} f_n'$?

(3) If (1) is true, does $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$?

Pointwise convergence

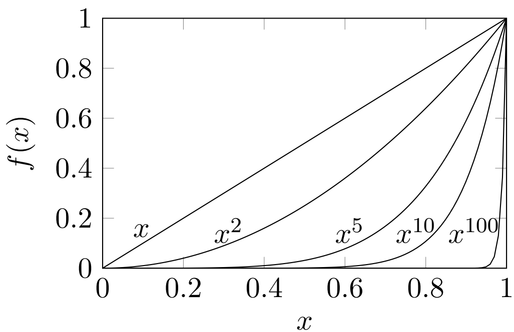
Definition 8.8 Let $(f_n)_{n=1}^{\infty}$ with $f_n \colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be a sequence of functions, and let $f \colon S \to \mathbf{R}$ be a function. We say that $(f_n)_{n=1}^{\infty}$ **converges pointwise** (or just **converges**) to f if for all $x \in S$, we have $\lim_{n\to\infty} f_n(x) = f(x)$.

Example 8.9 Let $f_n(x) = x^n$ be defined on $[0, 1]$, then we have the sequence of functions $(f_n)_{n=1}^{\infty}$ converges pointwise to $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

proof:

- if $x \in [0, 1)$: $\lim_{n\to\infty} x^n = 0$
- if $x = 1$: $\lim_{n\to\infty} 1^n = 1$

Remark 8.10 A sequence of continuous functions may not converge pointwise to a continuous function.



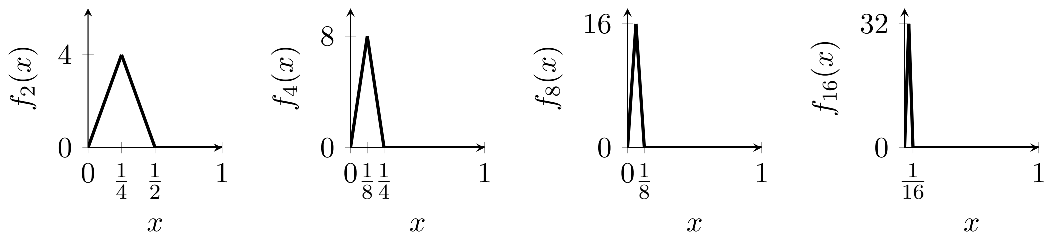
Example 8.11 Let $f_n(x)\colon [0,1] \to \mathbf{R}$ be defined by

$$f_n(x) = \begin{cases} 4n^2x & x \in \left[0, \frac{1}{2n}\right] \\ 4n - 4n^2x & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & x \in \left[\frac{1}{n}, 1\right], \end{cases}$$

then $(f_n)_{n=1}^{\infty}$ converges pointwise to $f(x) = 0$ $\big(x \in [0,1]\big)$.

proof: if $x = 0$, we have $\lim_{n\to\infty} f_n(0) = 0$; if $x \in (0,1]$, then $\exists M \in [0,1]$ such that $\forall n \geq M$, $\frac{1}{n} < x$, and hence,

$$(f_n(x))_{n=1}^{\infty} = f_1(x), \ldots, f_{M-1}(x), 0, 0, 0, \ldots \implies \lim_{n\to\infty} f_n(x) = 0$$



Uniform convergence

Definition 8.12 Let $(f_n)_{n=1}^{\infty}$ with $f_n\colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be a sequence of functions, and let $f\colon S \to \mathbf{R}$ be a function. We say that $(f_n)_{n=1}^{\infty}$ **converges uniformly** to f if for all $\epsilon > 0$, there exists some $M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, we have $|f_n(x) - f(x)| < \epsilon$.

Theorem 8.13 Let $f\colon S \to \mathbf{R}$, $f_n\colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be functions. If the sequence of functions $(f_n)_{n=1}^{\infty}$ converges uniformly to f, then $(f_n)_{n=1}^{\infty}$ converges pointwise to f.

---

proof: let $c \in S$, $\epsilon > 0$

- $(f_n)_{n=1}^{\infty}$ converges uniformly to $f \implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, $|f_n(x) - f(x)| < \epsilon$
- hence, $\forall n \geq M$, $|f_n(c) - f(c)| < \epsilon \implies (f_n)_{n=1}^{\infty}$ converges pointwise to f

---

Remark 8.14 Let $f\colon S \to \mathbf{R}$, $f_n\colon S \to \mathbf{R}$ for all $n \in \mathbf{N}$ be functions. The sequence $(f_n)_{n=1}^{\infty}$ does not converge to f uniformly if there exists some $\epsilon > 0$ such that for all $M \in \mathbf{N}$, there exist some $n \geq M$ and some $x \in S$, so that $|f_n(x) - f(x)| \geq \epsilon$.

Theorem 8.15 Let $f_n(x) = x^n$, $n \in \mathbf{N}$, and let $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$.

- The sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $[0,b]$ for all $0 < b < 1$.
- The sequence $(f_n)_{n=1}^{\infty}$ does not converges to f uniformly on $[0,1]$.

proof:

- let $\epsilon > 0$, $b \in (0,1)$, then $b^n \to 0 \implies \exists M \in \mathbf{N}$ such that $\forall n \geq M$, $b^n < \epsilon \implies \forall n \geq M$ and $x \in [0,b]$, we have

$$|f_n(x) - f(x)| = x^n \leq b^n < \epsilon$$

- choose $\epsilon = 1/2$, then $\forall M \in \mathbf{N}$, choose $n = M$, $x = (1/2)^{1/M} < 1$, we have

$$|f_M(x) - f(x)| = x^M = 1/2 \geq \epsilon$$

Interchange of limits

Example 8.16 In general, limits cannot be interchanged. For example,

$$\lim_{n\to\infty}\lim_{k\to\infty}\frac{n/k}{n/k+1}=\lim_{n\to\infty}0=0,\qquad \lim_{k\to\infty}\lim_{n\to\infty}\frac{n/k}{n/k+1}=\lim_{k\to\infty}1=1.$$

Remark 8.17 Based on example 8.16, we may ask the following questions.

- If $f_n\colon S\to\mathbf{R}$ with f_n continuous for all $n\in\mathbf{N}$ and $(f_n)_{n=1}^{\infty}$ converges to f uniformly or pointwise, then is f continuous?
- If $f_n\colon [a,b]\to\mathbf{R}$ with f_n differentiable for all $n\in\mathbf{N}$, and $(f_n)_{n=1}^{\infty}$ converges to f, $(f_n')_{n=1}^{\infty}$ converges to g uniformly or pointwise, then is f differentiable and does $f'=g$?
- If $f_n\colon [a,b]\to\mathbf{R}$, $n\in\mathbf{N}$, $f\colon [a,b]\to\mathbf{R}$, with f_n and f continuous, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly or pointwise, then does $\int_a^b f=\lim_{n\to\infty}\int_a^b f_n$?

Remark 8.18 If convergence is only pointwise, the answer is *no* for all questions in remark 8.17.

- Let $f_n(x) = x^n$ on $[0,1]$, $n \in \mathbf{N}$. Example 8.9 shows that $(f_n)_{n=1}^{\infty}$ converges pointwise to a noncontinuous function.

- Let $f_n(x) = \frac{x^{n+1}}{n+1}$ on $[0,1]$, then $(f_n)_{n=1}^{\infty}$ converges to $f(x) = 0$ pointwise on $[0,1]$ and $(f_n')_{n=1}^{\infty}$ converges pointwise to g given by $g(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$, but $f'(1) = 0 \neq g(1) = 1$.

- Let $f_n \colon [0,1] \to \mathbf{R}$ be given by $f_n(x) = \begin{cases} 4n^2 x & x \in \left[0, \frac{1}{2n}\right] \\ 4n - 4n^2 x & x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & x \in \left[\frac{1}{n}, 1\right] \end{cases}$, then $(f_n)_{n=1}^{\infty}$ converges to $f(x) = 0$ pointwise on $[0,1]$ (example 8.11), but

$$\int_0^1 f = 0 \neq \lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} \left(\frac{1}{2} \cdot \frac{1}{n} \cdot 2n\right) = 1.$$

Theorem 8.19 If $f_n\colon S \to \mathbf{R}$ is continuous for all $n \in \mathbf{N}$, $f\colon S \to \mathbf{R}$, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly, then f is continuous.

---

proof: let $c \in S$, $\epsilon > 0$

- f_n continuous on S, $c \in S \implies \exists \delta > 0$ such that for all $x \in S$ and $|x - c| < \delta$, we have $|f_n(x) - f_n(c)| < \epsilon/3$

- $f_n \to f$ uniformly $\implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in S$, we have $|f(x) - f_n(x)| < \epsilon/3$

- hence, for all $x \in S$ and $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_M(x) + f_M(x) - f_M(c) + f_M(c) - f(c)| \\ &\leq |f(x) - f_M(x)| + |f_M(x) - f_M(c)| + |f_M(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Theorem 8.20 If $f_n\colon [a,b] \to \mathbf{R}$ is continuous for all $n \in \mathbf{N}$, $f\colon [a,b] \to \mathbf{R}$, and $(f_n)_{n=1}^{\infty}$ converges to f uniformly, then $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

proof: let $\epsilon > 0$

- by theorem 8.19, we know that f is continuous on $[a,b]$
- $(f_n)_{n=1}^{\infty}$ converges uniformly to $f \implies \exists M \in \mathbf{N}$ such that for all $n \geq M$ and $x \in [a,b]$, we have $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$
- hence, for all $n \geq M$, we have

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f| < \int_a^b \frac{\epsilon}{b-a} = \epsilon,$$

  where the first inequality is by corollary 7.17

Remark 8.21 Notationally, theorem 8.20 says that

$$\int_a^b f = \int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n.$$

Theorem 8.22 If $f_n\colon [a,b] \to \mathbf{R}$ is continuously differentiable, $f\colon [a,b] \to \mathbf{R}$, $g\colon [a,b] \to \mathbf{R}$, and

- $(f_n)_{n=1}^{\infty}$ converges to f pointwise,
- $(f_n')_{n=1}^{\infty}$ converges to g uniformly,

then f is continuously differentiable and $f' = g$.

proof: let $x \in [a,b]$

- by theorem 8.19, we know that g is continuous on $[a,b]$
- by theorem 7.19, we have

$$\int_a^x f_n' = f_n(x) - f(a) \implies \lim_{n\to\infty} \int_a^x f_n' = \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a)$$

- $f_n \to f$ pointwise $\implies \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a) = f(x) - f(a)$
- $f_n' \to g$ uniformly $\implies \lim_{n\to\infty} \int_a^x f_n' = \int_a^x g$ (theorem 8.20)
- put together, we have

$$\int_a^x g = f(x) - f(a) \implies \left(\int_a^x g\right)' = g(x) = f'(x)$$

Weierstrass M-test

Theorem 8.23 *Weierstrass M-test.* Let $f_k\colon S \to \mathbf{R}$ for all $k \in \mathbf{N}$. Suppose there exists $M_k > 0$, $k \in \mathbf{N}$, such that

(a) $|f_k(x)| \le M_k$ for all $x \in S$,

(b) $\sum_{k=1}^{\infty} M_k$ converges.

Then, we have the following conclusion.

(1) The series $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely for all $x \in S$.

(2) Let $f(x) = \sum_{k=1}^{\infty} f_k(x)$ for all $x \in S$, then the series $(\sum_{k=1}^{n} f_k)_{n=1}^{\infty}$ converges to f uniformly on S.

---

proof:

(1) $|f_k(x)| \le M_k$, $\sum_{k=1}^{\infty} M_k$ converges $\implies$ $\sum_{k=1}^{\infty} |f_k(x)|$ converges (theorem 4.20) $\implies$ $\sum_{k=1}^{\infty} f_k(x)$ converges absolutely

(2) let $\epsilon > 0$; $\sum_{k=1}^{\infty} M_k$ converges $\implies$ $\exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, we have

$$\sum_{k=n+1}^{\infty} M_k = \left| \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k \right| < \epsilon$$

then, for all $n \geq M$ and $x \in S$, we have

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x) \right| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$$

Properties of power series

Theorem 8.24 Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series with radius of convergence $\rho \in (0,\infty]$, then for all $r \in (0,\rho)$, the series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges uniformly on $[x_0-r, x_0+r]$.

---

proof:

- note that we have $|x-x_0| \leq r$ for all $x \in [x_0-r, x_0+r]$
- let $f_k = a_k(x-x_0)^k$, choose $M_k = |a_k|r^k$, $k \in \mathbf{N}$, then $\forall x \in [x_0-r, x_0+r]$,

$$|f_k(x)| = |a_k(x-x_0)^k| = |a_k||x-x_0|^k \leq |a_k|r^k = M_k$$

- consider the root test (theorem 4.26) for $\sum_{k=0}^{\infty} M_k$, we have

$$L = \lim_{k\to\infty} M_k^{1/k} = \lim_{k\to\infty} \left(|a_k|r^k\right)^{1/k} = \lim_{k\to\infty} |a_k|^{1/k} r = \begin{cases} r/\rho & \rho < \infty \\ 0 & \rho = \infty \end{cases}$$

  since $r \in (0,\rho)$, we have $L < 1 \implies \sum_{k=0}^{\infty} M_k$ converges absolutely
- put together, by theorem 8.23, we have $\left(\sum_{k=0}^{n} f_k\right)_{n=1}^{\infty} = \sum_{k=0}^{n} a_k(x-x_0)^k$ converges uniformly on $[x_0-r, x_0+r]$

Theorem 8.25 Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series with radius of convergence $\rho \in (0, \infty]$, then we have the following conclusion.

- For all $c \in (x_0-\rho, x_0+\rho)$, the function given by the series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is differentiable at c, and

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k(x-x_0)^k\right)\Bigg|_{x=c} = \sum_{k=0}^{\infty} \frac{d}{dx}(a_k(x-x_0)^k)\Big|_{x=c}.$$

- For all a, b such that $x_0-\rho < a < b < x_0+\rho$,

$$\int_a^b \sum_{k=0}^{\infty} a_k(x-x_0)^k \, dx = \sum_{k=0}^{\infty} \int_a^b a_k(x-x_0)^k \, dx.$$