- cluster points of sets
- limits of functions and sequential properties
- left and right limits
- continuous functions
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## **Cluster points of sets**

**Definition 5.1** Let  $S \subseteq \mathbf{R}$ . We say that the point  $c \in \mathbf{R}$  is a **cluster point** of S if for all  $\delta > 0$ , we have  $(c - \delta, c + \delta) \cap S \setminus \{c\} \neq \emptyset$ , *i.e.*, for all  $\delta > 0$ , there exists some  $x \in S$ , such that  $0 < |x - c| < \delta$ .

### examples:

- $S = \{1/n \mid n \in \mathbf{N}\}$  has a cluster point c = 0
- S = (0,1) has a set of cluster points given by [0,1]
- +  $S={\bf Q}$  has a set of cluster points given by  ${\bf R}$
- $S = \{0\}$  has no cluster points
- $S = \mathbf{Z}$  has no cluster points

**Theorem 5.2** Let  $S \subseteq \mathbf{R}$ . Then c is a cluster point of S if and only if there exists a sequence  $(x_n)_{n=1}^{\infty}$  of elements in  $S \setminus \{c\}$  such that  $\lim_{n \to \infty} x_n = c$ .

### proof:

- suppose c is a cluster point of S, then  $\forall \delta > 0$ ,  $\exists x \in S$  such that  $0 < |x c| < \delta$ -  $\forall n \in \mathbf{N}$ , choose  $x_n \in S$  such that  $0 < |x_n - c| < \frac{1}{n}$ 
  - $-\frac{1}{n} \to 0 \implies |x_n c| \to 0 \implies x_n \to c$
- suppose there exists a sequence  $(x_n)_{n=1}^{\infty}$  with  $x_n \in S \setminus \{c\}$  for all  $n \in \mathbb{N}$  such that  $x_n \to c$ , let  $\delta > 0$

 $- \ x_n \to c \text{ with } x_n \in S \setminus \{c\} \implies \exists M \in \mathbf{N} \text{ such that } \forall n \geq M, \ 0 < |x_n - c| < \delta$ 

– choose  $x = x_M$ , then we have  $0 < |x - c| < \delta \implies S$  has cluster point c

## Limits of functions

**Definition 5.3** Let  $f: S \to \mathbf{R}$  be a function and c be a cluster point of  $S \subseteq \mathbf{R}$ . Suppose there exists an  $L \in \mathbf{R}$ , and for all  $\epsilon > 0$ , there exists some  $\delta > 0$  such that for all  $x \in S$  and  $0 < |x - c| < \delta$ , we have  $|f(x) - L| < \epsilon$ . We then say f(x) converges to L as x goes to c, and we write

 $f(x) \to L$  as  $x \to c$ .

We say L is a **limit** of f(x) as x goes to c, and if L is unique, we write

$$\lim_{x \to c} f(x) = L.$$

**Remark 5.4** The function  $f: S \to \mathbf{R}$  does not converge to  $L \in \mathbf{R}$  as x goes to a cluster point c of S implies that there exists some  $\epsilon > 0$ , such that for all  $\delta > 0$ , there exists some  $x \in S$  and  $0 < |x - c| < \delta$ , so that  $|f(x) - L| \ge \epsilon$ .

**Theorem 5.5** Let  $f: S \to \mathbf{R}$  be a function and c be a cluster point of  $S \subseteq \mathbf{R}$ . If  $f(x) \to L_1$  and  $f(x) \to L_2$  as  $x \to c$ , then  $L_1 = L_2$ .

**proof:** let  $\epsilon > 0$ 

- $f(x) \to L_1$  as  $x \to c \implies \exists \delta_1 > 0$  such that for all  $x \in S$  and  $0 < |x c| < \delta_1$ ,  $|f(x) L_1| < \epsilon/2$
- $f(x) \to L_2$  as  $x \to c \implies \exists \delta_2 > 0$  such that for all  $x \in S$  and  $0 < |x c| < \delta_2$ ,  $|f(x) L_2| < \epsilon/2$
- choose  $\delta = \min\{\delta_1, \delta_2\}$ , then for all  $x \in S$  and  $0 < |x c| < \delta$ , we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \le |f(x) - L_1| + |f(x) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$$
  
$$\implies L_1 = L_2$$

**Example 5.6** Let f(x) = ax + b. Then, for all  $c \in \mathbf{R}$ , we have  $\lim_{x\to c} f(x) = ac + b$ .

**proof:** let  $\epsilon > 0$ , choose  $\delta = \frac{\epsilon}{|a|+1}$ , then for all  $x \in \mathbf{R}$  and  $0 < |x - c| < \delta$ , we have

$$|f(x) - (ac + b)| = |(ax + b) - (ac + b)| = |a||x - c| < |a|\delta = \frac{|a|}{|a| + 1}\epsilon \le \epsilon$$

**Example 5.7** Let  $f: (0, \infty) \to \mathbf{R}$  with  $f(x) = \sqrt{x}$ . Then, for all c > 0, we have  $\lim_{x\to c} f(x) = \sqrt{c}$ .

**proof:** let  $\epsilon > 0$ , choose  $\delta = \epsilon \sqrt{c}$ , then for all x > 0 and  $0 < |x - c| < \delta$ , we have

$$|f(x) - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| = \left|\frac{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})}{\sqrt{x} + \sqrt{c}}\right| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| \le \frac{|x - c|}{\sqrt{c}} < \frac{\delta}{\sqrt{c}} < \epsilon$$

**Example 5.8** Let 
$$f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$$
. Then,  $\lim_{x \to 0} f(x) = 1 \ (\neq f(0))$ .

**proof:** let  $\epsilon > 0$ , choose  $\delta = 1$ , then  $\forall x$  satisfies  $0 < |x| < \delta$ , we have  $x \neq 0 \implies \forall x$  satisfies  $0 < |x| < \delta$ , we have  $|f(x) - 1| = |1 - 1| = 0 < \epsilon$ 

**Theorem 5.9** Let  $f: S \to \mathbf{R}$  be a function and c be a cluster point of  $S \subseteq \mathbf{R}$ . Then, the following statements are equivalent:

- The function f(x) converges to  $L \in \mathbf{R}$  as x goes to c, *i.e.*,  $\lim_{x \to c} f(x) = L$ .
- For all sequences  $(x_n)_{n=1}^{\infty}$  in  $S \setminus \{c\}$  such that  $\lim_{n \to \infty} x_n = c$ , we have  $\lim_{n \to \infty} f(x_n) = L$ .

### proof:

• suppose 
$$\lim_{x\to c} f(x) = L$$
, let  $\epsilon > 0$   
-  $\exists \delta > 0$ , such that for all  $x \in S$  and  $0 < |x - c| < \delta$ , we have  $|f(x) - L| < \epsilon$ 

- $\begin{array}{l} \ x_n \to c, \ x_n \in S \setminus \{c\} \implies \exists M \in \mathbf{N} \text{ such that } \forall n \geq M, \ 0 < |x_n c| < \delta \implies \forall n \geq M, \text{ we have } |f(x_n) L| < \epsilon, \ i.e., \ f(x_n) \to L \end{array}$
- suppose for all sequences in  $S \setminus \{c\}$  s.t.  $x_n \to c$ , we have  $f(x_n) \to L$ 
  - assume  $\lim_{x\to c} f(x) \neq L \implies \exists \epsilon > 0 \text{ s.t. } \forall \delta > 0$ , there exists some  $x \in S$  and  $0 < |x c| < \delta$ , so that  $|f(x) L| \geq \epsilon$
  - choose a sequence  $(x_n)_{n=1}^{\infty}$  s.t.  $\forall n \in \mathbb{N}$ ,  $x_n \in S \setminus \{c\}$ ,  $0 < |x_n c| < \frac{1}{n}$ , and  $|f(x_n) L| \ge \epsilon$  for all  $n \in \mathbb{N}$
  - however,  $\frac{1}{n} \to 0 \implies x_n \to c \implies f(x_n) \to L \implies \exists M \in \mathbf{N} \text{ s.t. } \forall n \ge M$ ,  $|f(x_n) L| < \epsilon$ , which is a contradiction

**Theorem 5.10** For all  $c \in \mathbf{R}$ , we have  $\lim_{x\to c} x^2 = c^2$ .

**proof:** let  $(x_n)_{n=1}^{\infty}$  be a sequence in  $\mathbf{R} \setminus \{c\}$  such that  $x_n \to c$ , then according to theorem 3.24, we have  $x_n^2 \to c^2 \implies \lim_{x\to c} x^2 = c^2$  (theorem 5.9)

**Theorem 5.11** The limit  $\lim_{x\to 0} \sin(1/x)$  does not exist, but  $\lim_{x\to 0} x \sin(1/x) = 0$ .

#### proof:

- we first show that  $\lim_{x\to 0} x \sin(1/x) = 0$ : let  $(x_n)_{n=1}^{\infty}$  be a sequence in  $\mathbf{R} \setminus \{0\}$  such that  $x_n \to 0$ ; since  $0 \le |x_n \sin(1/x_n)| \le |x_n|$  for all  $n \in \mathbf{N}$ , and  $x_n \to 0$ , we have  $|x_n \sin(1/x_n)| \to 0 \implies \lim_{x\to 0} x \sin(1/x) = 0$
- we now show that  $\lim_{x\to 0} \sin(1/x)$  does not exist:
  - choose a sequence  $(x_n)_{n=1}^{\infty}$  where  $x_n = \frac{2}{(2n-1)\pi}$ , then we have  $x_n \to 0$
  - consider the sequence  $(\sin(1/x_n))_{n=1}^{\infty}$ , we have

$$\sin(1/x_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n+1}$$

 $\implies (\sin(1/x_n))_{n=1}^{\infty}$  does not converge  $\implies \lim_{x \to 0} \sin(1/x)$  does not exist

## Sequential properties

**Theorem 5.12** Let  $f,g: S \to \mathbf{R}$  be functions and c be a cluster point of  $S \subseteq \mathbf{R}$ . Suppose  $f(x) \leq g(x)$  for all  $x \in S$ , and we have  $\lim_{x\to c} f(x)$  and  $\lim_{x\to c} g(x)$  both exist, then  $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$ .

**proof:** let  $(x_n)_{n=1}^{\infty}$  be a sequence in  $S \setminus \{c\}$  such that  $x_n \to c$ 

- $\lim_{x\to c} f(x)$  and  $\lim_{x\to c} g(x)$  exist  $\implies (f(x_n))_{n=1}^{\infty}$  and  $(g(x_n))_{n=1}^{\infty}$  converges
- let  $f(x_n) \to L_1$ ,  $g(x_n) \to L_2$ , since  $f(x) \le g(x)$  for all  $x \in S$ , we have  $L_1 \le L_2$ , *i.e.*,  $\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$

similarly, we can prove the following theorems using the properties of sequences:

**Theorem 5.13** Let  $f: S \to \mathbf{R}$  be a function and c be a cluster point of  $S \subseteq \mathbf{R}$ . Suppose the limit  $\lim_{x\to c} f(x)$  exists, and there exists  $a, b \in \mathbf{R}$  such that  $a \leq f(x) \leq b$  for all  $x \in S \setminus \{c\}$ , then  $a \leq \lim_{x\to c} f(x) \leq b$ . **Theorem 5.14** Let c be a cluster point of  $S \subseteq \mathbf{R}$ , and  $f, g, h: S \to \mathbf{R}$  be functions such that  $f(x) \leq g(x) \leq h(x)$  for all  $x \in S \setminus \{c\}$ . Suppose  $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$ , then  $\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$ .

**Theorem 5.15** Let c be a cluster point of  $S \subseteq \mathbf{R}$ , and  $f, g: S \to \mathbf{R}$  be functions such that  $\lim_{x\to c} f(x)$  and  $\lim_{x\to c} g(x)$  both exist, we have:

• 
$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x);$$

- $\lim_{x\to c} (f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x);$
- if  $\lim_{x\to c} g(x) \neq 0$  and  $g(x) \neq 0$  for all  $x \in S \setminus \{c\}$ , then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)};$$

**Theorem 5.16** Let c be a cluster point of  $S \subseteq \mathbf{R}$  and  $f: S \to \mathbf{R}$  be a function such that  $\lim_{x\to c} f(x)$  exists, then we have  $\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)|$ .

## Left and right limits

**Definition 5.17** Let  $S \subseteq \mathbf{R}$  and  $f: S \to \mathbf{R}$  be a function.

Suppose c is a cluster point of  $S \cap (-\infty, c)$ , we say f(x) converges to L as  $x \to c^-$ , if for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in S$  and  $c - \delta < x < c$ , we have  $|f(x) - L| < \epsilon$ . We call such a limit the **left limit** of f at c, denoted  $\lim_{x\to c^-} f(x)$ .

Suppose c is a cluster point of  $S \cap (c, \infty)$ , we say f(x) converges to L as  $x \to c^+$ , if for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in S$  and  $c < x < c + \delta$ , we have  $|f(x) - L| < \epsilon$ . We call such a limit the **right limit** of f at c, denoted  $\lim_{x\to c^+} f(x)$ .

**Example 5.18** Consider the function f given by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0, \end{cases}$$

we have  $\lim_{x\to 0^-} f(x) = 0$  and  $\lim_{x\to 0^+} f(x) = 1$ , even if f(0) is undefined.

## **Continuous functions**

**Definition 5.19** Let  $S \subseteq \mathbf{R}$  and  $c \in S$ . We say the function f is **continuous** at c if for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in S$  and  $|x - c| < \delta$ , we have  $|f(x) - f(c)| < \epsilon$ .

We say the function f is continuous on the set U for  $U \subseteq S$  if f is continuous at every point of U.

**Remark 5.20** The function f is not continuous at point  $c \in S$  if there exists some  $\epsilon > 0$  such that for all  $\delta > 0$ , there exists some  $x \in S$  and  $|x - c| < \delta$ , so that  $|f(x) - f(c)| \ge \epsilon$ .

**Example 5.21** The function f(x) = ax + b is continuous on **R**.

**proof:** let  $c \in \mathbf{R}$ ,  $\epsilon > 0$ , choose  $\delta = \frac{\epsilon}{|a|+1}$ , then for all  $x \in \mathbf{R}$  and  $|x - c| < \delta$ , we have

$$|f(x) - f(c)| = |ax + b - ac - b| = |a||x - c| < |a|\delta = \frac{|a|}{|a| + 1}\epsilon \le \epsilon$$

**Example 5.22** The function f given by

$$f(x) = \begin{cases} 1 & x \neq 0\\ 2 & x = 0 \end{cases}$$

is not continuous at c = 0.

**proof:** choose  $\epsilon = 1$  and let  $\delta > 0$ , then  $x = \delta/2$  satisfies  $|x| < \delta$ , but

$$|f(x) - f(0)| = |1 - 0| = 1 \ge \epsilon$$

**Theorem 5.23** Let  $S \subseteq \mathbf{R}$  be a set,  $c \in S$  be a point, and  $f: S \to \mathbf{R}$  be a function.

- If c is not a cluster point of S, then the function f is continuous at c.
- If c is a cluster point of S, then the function f is continuous at c if and only if
   lim<sub>x→c</sub> f(x) = f(c).
- The function f is continuous at c if and only if for all sequences (x<sub>n</sub>)<sub>n=1</sub><sup>∞</sup> in S with lim<sub>n→∞</sub> x<sub>n</sub> = c, we have lim<sub>n→∞</sub> f(x<sub>n</sub>) = f(c).

**proof:** to show the first statement, let  $\epsilon > 0$ 

- $c \in S$  and c is not a cluster point of  $S \implies \exists \delta > 0$  s.t.  $(c \delta, c + \delta) \cap S = \{c\}$
- then for all  $x \in S$  such that  $|x c| < \delta$ , we have x = c, and hence,

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

we now show the second statement:

- suppose f is continuous at c, let  $\epsilon > 0$ 
  - f is continuous at  $c\implies \exists \delta>0$  such that for all  $x\in S$  and  $|x-c|<\delta,$  we have  $|f(x)-f(c)|<\epsilon$
  - then  $\forall x \in S \text{ s.t. } 0 < |x c| < \delta$ ,  $|f(x) f(c)| < \epsilon \implies \lim_{x \to c} f(x) = f(c)$

- suppose  $\lim_{x\to c} f(x) = f(c)$ , let  $\epsilon > 0$ 
  - $f(x) \to f(c)$  as  $x \to c \implies \exists \delta > 0$  such that for all  $x \in S$  and  $0 < |x c| < \delta$ , we have  $|f(x) f(c)| < \epsilon$
  - then for all  $x\in S$  such that  $|x-c|<\delta :$  if x=c, we have

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$$

 $\text{ if } x \neq c \text{, we have } 0 < |x-c| < \delta \implies |f(x)-f(c)| < \epsilon$ 

– put together, we conclude that the function f is continuous at  $\boldsymbol{c}$ 

we now show the third statement

- suppose f is continuous at c, let  $(x_n)_{n=1}^{\infty}$  be a sequence in S,  $x_n \to c$ , let  $\epsilon > 0$ - f is continuous at  $c \implies \exists \delta > 0$  such that for all  $x \in S$  and  $|x - c| < \delta$ , we have  $|f(x) - f(c)| < \epsilon$ 
  - $\begin{array}{l} -x_n \to c \implies \exists M \in \mathbf{N} \text{ such that } \forall n \geq M, \ |x_n c| < \delta \implies \forall n \geq M, \\ |f(x_n) f(c)| < \epsilon \implies (f(x_n))_{n=1}^{\infty} \to f(c) \end{array}$
- suppose for all  $(x_n)_{n=1}^{\infty}$  in S such that  $x_n \to c$ , we have  $f(x_n) \to f(c)$ 
  - assume f is not continuous at  $c\implies \exists\epsilon>0,\,\forall\delta>0,\,\exists x\in S$  such that  $|x-c|<\delta,$  but  $|f(x)-f(c)|\geq\epsilon$
  - choose  $x_n \in S$  such that  $\forall n \in \mathbf{N}, \ 0 \le |x_n c| < \frac{1}{n}$  but  $|f(x_n) f(x)| \ge \epsilon$

$$\begin{array}{l} - \ \frac{1}{n} \to 0 \implies x_n \to c \implies f(x_n) \to f(c) \implies \exists M \in \mathbf{N} \text{ such that } \forall n \geq M, \\ |f(x_n) - f(c)| < \epsilon, \text{ which is a contradiction} \end{array}$$

**Theorem 5.24** The functions  $\sin x$  and  $\cos x$  are continuous functions on **R**.

### proof:

• recall the following properties of  $\sin x$  and  $\cos x$  for all  $x \in \mathbf{R}$ :

$$-\sin^2(x) + \cos^2(x) = 1 \implies |\sin x| \le 1 \text{ and } |\cos x| \le 1$$

- $|\sin x| \le |x|$
- $-\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$
- $-\sin(a) \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$
- we first show that  $\sin x$  is continuous, let  $c \in \mathbf{R}$ , let  $\epsilon > 0$ , choose  $\delta = \epsilon$ , then for all  $x \in \mathbf{R}$  such that  $|x c| < \delta$ , we have

$$\left|\sin x - \sin c\right| = \left|2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)\right| \le 2\left|\sin\left(\frac{x-c}{2}\right)\right| \le 2\frac{|x-c|}{2} = |x-c| < \epsilon$$

• we now show that  $\cos x$  is continuous, let  $c \in \mathbf{R}$ , let  $(x_n)_{n=1}^{\infty}$  be a sequence with  $x_n \to c$ , then we have  $x_n + \frac{\pi}{2} \to c + \frac{\pi}{2}$ , and hence,

$$\lim_{n \to \infty} \cos x_n = \lim_{n \to \infty} \sin \left( x_n + \frac{\pi}{2} \right) = \sin \left( c + \frac{\pi}{2} \right) = \cos c$$

**Theorem 5.25** *Dirichlet function.* The Dirichlet function given by

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

is not continuous on all of  $\mathbf{R}$ .

**proof:** let  $c \in \mathbf{R}$ 

• if  $c \in \mathbf{Q}$ , then for all  $n \in \mathbf{N}$ , there exists  $x_n \notin \mathbf{Q}$  such that  $c < x_n < c + \frac{1}{n}$ ;  $\frac{1}{n} \to 0 \implies x_n \to c$ , however,

$$\lim_{n \to \infty} f(x_n) = 0 \neq f(c) = 1$$

 $\implies (f(x_n))_{n=1}^\infty$  does not converge to f(c)

• if  $c \notin \mathbf{Q}$ , then for all  $n \in \mathbf{N}$ , there exists  $x_n \in \mathbf{Q}$  such that  $c < x_n < c + \frac{1}{n}$ ;  $\frac{1}{n} \to 0 \implies x_n \to c$ , however,

$$\lim_{n \to \infty} f(x_n) = 1 \neq f(c) = 0$$

$$\implies (f(x_n))_{n=1}^{\infty}$$
 does not converge to  $f(c)$ 

## Operations that preserves continuity

**Theorem 5.26** Let  $f, g: S \to \mathbf{R}$  be functions on  $S \subseteq \mathbf{R}$  and are continuous at  $c \in S$ .

- The function f + g is continuous at c.
- The function  $f \cdot g$  is continuous at c.
- If  $g(x) \neq 0$  for all  $x \in S$ , then the function f/g is continuous at c.

**proof:** we show that the function f + g is continuous at c, the other two statements can be proved similarly; let  $(x_n)_{n=1}^{\infty}$  be a sequence in S with  $x_n \to c$ 

- f is continuous at  $c \implies \lim_{n \to \infty} f(x_n) = f(c)$
- g is continuous at  $c \implies \lim_{n \to \infty} g(x_n) = g(c)$
- hence,  $\lim_{n\to\infty}(f(x_n) + g(x_n)) = f(c) + g(c) \implies f + g$  is continuous at c

**Theorem 5.27** Let  $f: B \to \mathbf{R}$  and  $g: A \to B$  be functions on  $A, B \subseteq \mathbf{R}$ . If g is continuous at  $c \in A$  and f is continuous at  $g(c) \in B$ , then  $f \circ g$  is continuous at c.

**proof:** let  $(x_n)_{n=1}^{\infty}$  be a sequence in A and  $x_n \to c \implies g(x_n) \to g(c) \implies f(g(x_n)) \to f(g(c)) \implies f \circ g$  is continuous at c

**Theorem 5.28** Let f be a polynomial function of the form

$$f(x) = a_p x^p + \dots + a_1 x + a_0.$$

Then, the function f is continuous on  $\mathbf{R}$ .

**proof:** let  $c \in \mathbf{R}$ , let  $(x_n)_{n=1}^{\infty}$  be a sequence in  $\mathbf{R}$  and  $x_n \to c$ , then we have

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} (a_p x_n^p + \dots + a_1 x_n + a_0)$$
$$= a_p \lim_{n \to \infty} x_n^p + \dots + a_1 \lim_{n \to \infty} x_n + a_0$$
$$= a_p c^p + \dots + a_1 c + a_0 = f(c)$$

**Example 5.29** Theorems 5.26 and 5.27 allows us to show that some given function is continuous without a huge  $\epsilon - \delta$  proof, for example:

- The function  $1/x^2$  is continuous on  $(0,\infty)$ , since  $x^2$  is continuous on  $(0,\infty)$ .
- The function  $(\cos(1/x^2))^2$  is continuous on  $(0, \infty)$ , since  $\cos x$  is continuous on **R**, and  $x^2$  is continuous on  $(0, \infty)$ .

### Extreme value theorem

**Definition 5.30** A function  $f: S \to \mathbf{R}$  is **bounded** if there exists some  $B \ge 0$  such that for all  $x \in S$ , we have  $|f(x)| \le B$ .

**Theorem 5.31** If the function  $f: [a, b] \rightarrow \mathbf{R}$  is continuous then f is bounded.

### proof:

- suppose f is unbounded, then  $\forall B \ge 0$ ,  $\exists x \in [a, b]$  such that |f(x)| > B
- let  $(x_n)_{n=1}^{\infty}$  be a sequence in [a, b] such that for all  $n \in \mathbf{N}$ ,  $|f(x_n)| > n$
- $(x_n)_{n=1}^{\infty}$  is in  $[a,b] \implies (x_n)_{n=1}^{\infty}$  is bounded  $\implies$  there exists a subsequence  $(x_{n_i})_{i=1}^{\infty}$  (theorem 3.37) that converges to  $c \in \mathbf{R}$
- $a \leq x_n \leq b \implies a \leq x_{n_i} \leq b \implies c \in [a, b]$
- f is continuous on  $[a,b] \implies f(x_{n_i}) \rightarrow f(c) \implies (f(x_{n_i}))_{i=1}^{\infty}$  is bounded
- however,  $|f(x_{n_i})|>n_i\implies (n_i)_{i=1}^\infty$  is bounded, which is a contradiction

**Definition 5.32** Let  $f: S \to \mathbf{R}$  be a function. We say the function f achieves an **absolute minimum** at c if  $f(x) \ge f(c)$  for all  $x \in S$ . We say the function f achieves an **absolute maximum** at d if  $f(x) \le f(d)$  for all  $x \in S$ .

**Theorem 5.33** Extreme value theorem. Let  $f: [a, b] \to \mathbf{R}$  be a function on a closed, bounded interval [a, b]. If the function f is continuous on [a, b], then f achieves absolute maximum and absolute minimum on [a, b].

proof: we show the case for absolute maximum

- f is continuous on  $[a, b] \implies f$  is bounded  $\implies$  the set  $E = \{f(x) \mid x \in [a, b]\}$  is bounded  $\implies \sup E \in \mathbf{R}$  exists
- $\sup E$  is the supremum of  $\{f(x) \mid x \in [a,b]\} \implies \forall x \in [a,b], f(x) \leq \sup E$ , and, there exists some sequence  $(f(x_n))_{n=1}^{\infty}$  with  $x_n \in [a,b]$  such that  $f(x_n) \to \sup E$
- $(x_n)_{n=1}^{\infty}$  is in  $[a,b] \implies$  there exists a subsequence  $(x_{n_i})_{i=1}^{\infty}$  such that  $x_{n_i} \to d$ and  $d \in [a,b] \implies f(x_{n_i}) \to f(d)$  (since f is continuous)
- $f(x_n) \to \sup E \implies f(x_{n_i}) \to \sup E \implies \sup E = f(d) \implies$  there exists a point  $d \in [a, b]$  such that  $f(x) \le f(d)$  for all  $x \in [a, b]$

**Remark 5.34** To apply the extreme value theorem, the function f has to be continuous on a closed, bounded interval.

If the function  $f \colon [a,b] \to \mathbf{R}$  is not continuous, consider the function given by

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \text{ or } x = 1\\ x & x \in (0, 1), \end{cases}$$

which neither achieves an absolute maximum nor an absolute minimum on [0, 1].

If the function  $f: S \to \mathbf{R}$  is continuous but S not closed and bounded, consider the function given by

$$f(x) = \frac{1}{x} - \frac{1}{1-x}, \quad S = (0,1),$$

which neither achieves an absolute maximum nor an absolute minimum on [0, 1].

### Intermediate value theorem

**Theorem 5.35** Let  $f: [a, b] \to \mathbf{R}$  be a continuous function. If f(a) < 0 and f(b) > 0, then there exists some  $c \in (a, b)$  such that f(c) = 0.

**proof:** let  $a_1 = a$ ,  $b_1 = b$ , for all  $n \in \mathbf{N}$ , given  $a_n$  and  $b_n$ , define  $a_{n+1}$  and  $b_{n+1}$  as:

- $a_{n+1} = a_n, \ b_{n+1} = \frac{a_n + b_n}{2}, \ \text{if} \ f\left(\frac{a_n + b_n}{2}\right) \ge 0$
- $a_{n+1} = \frac{a_n + b_n}{2}$ ,  $b_{n+1} = b_n$ , if  $f\left(\frac{a_n + b_n}{2}\right) < 0$

then the sequences  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  has the following properties:

- $a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$  for all  $n \in \mathbf{N} \implies (a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are monotone and bounded  $\implies (a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  converge, let  $a_n \to c$ ,  $b_n \to d$
- $f(a_n) \leq 0$ ,  $f(b_n) \geq 0$  for all  $n \in \mathbb{N}$ , since f is continuous,  $c, d \in [a, b] \implies \lim_{n \to \infty} f(a_n) = f(c) \leq 0$  and  $\lim_{n \to \infty} f(b_n) = f(d) \geq 0$

• 
$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} = \frac{b_{n-1} - a_{n-1}}{2^2} = \dots = \frac{b-a}{2^n} \implies b_n - a_n = \frac{1}{2^{n-1}}(b-a)$$
  
 $\implies \lim_{n \to \infty} (b_n - a_n) = \lim_{n \to \infty} \frac{1}{2^{n-1}}(b-a) = 0 = \lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n$   
 $\implies \lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n \implies c = d$ 

put together, we have  $f(c) \leq 0$ ,  $f(d) \geq 0$ , and  $f(c) = f(d) \implies f(c) = f(d) = 0$  $\implies \exists c \in (a, b) \text{ such that } f(c) = 0$ 

**Theorem 5.36** Bolzano's intermediate value theorem. Let  $f: [a, b] \to \mathbf{R}$  be a continuous function. Suppose  $y \in \mathbf{R}$  such that f(a) < y < f(b) or f(b) < y < f(a), then there exists a  $c \in (a, b)$  such that f(c) = y.

**proof:** we consider the case for f(a) < y < f(b), the other case is similar

- let  $g: [a,b] \to \mathbf{R}$  be a function given by g(x) = f(x) y, then g is continuous on [a,b] (theorem 5.26)
- $f(a) < y < f(b) \implies g(a) = f(a) y < 0, \ g(b) = f(b) y > 0 \implies \exists c \in (a, b)$ such that g(c) = f(c) - y = 0 (theorem 5.35)  $\implies \exists c \in (a, b)$  such that f(c) = y

**Theorem 5.37** Let  $f: [a, b] \to \mathbf{R}$  be a continuous function. Suppose the function f achieves absolute minimum at  $c \in [a, b]$ , and achieves absolute maximum at  $d \in [a, b]$ . Then, we have f([a, b]) = [f(c), f(d)], *i.e.*, every value between the absolute minimum value and the absolute maximum value is achieved.

### proof:

- according to theorem 5.33, we have  $f([a,b]) \subseteq [f(c), f(d)]$
- according to theorem 5.36, we have  $[f(c),f(d)]\subseteq f([c,d])\subseteq f([a,b])$
- hence,  $f([a,b]) = \left[f(c),f(d)\right]$

**Remark 5.38** Similarly, theorem 5.36 is false if the function f is not continuous.

**Example 5.39** The polynomial given by  $f(x) = x^{2021} + x^{2020} + 9.03x + 1$  has at least one real root.

**proof:** we have f(0) = 1 > 0 and f(-1) = -8.03 < 0, hence, by theorem 5.36, there exists some  $c \in (-1,0)$  such that f(c) = 0

## **Uniform continuity**

**Example 5.40** The function  $f(x) = \frac{1}{x}$  is continuous on (0, 1).

**proof:** let  $c \in (0,1)$  and  $\epsilon > 0$ , choose  $\delta = \min\left\{\frac{c}{2}, \frac{c^2}{2}\epsilon\right\}$ , then  $\forall x \in (0,1)$  such that  $|x-c| < \delta$ , we have

• 
$$||x| - |c|| \le |x - c| < \delta \le \frac{c}{2} \implies -\frac{c}{2} < |x| - c \implies \frac{1}{|x|} < \frac{2}{c}$$

• hence, 
$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x|c} < \frac{\delta}{|x|c} < \frac{2\delta}{c^2} \le \frac{2}{c^2} \cdot \frac{c^2}{2}\epsilon = \epsilon$$

**Remark 5.41** Example 5.40 shows that in the definition of function continuity, the number  $\delta$  can depend on both the number  $\epsilon$  and the point c.

**Definition 5.42** Let  $f: S \to \mathbf{R}$  be a function. We say the function f is **uniformly continuous** on S if for all  $\epsilon > 0$ , there exists some  $\delta > 0$  such that for all  $x, c \in S$  and  $|x-c| < \delta$ , we have  $|f(x) - f(c)| < \epsilon$ .

**Remark 5.43** In the definition of uniform continuity, the number  $\delta$  only depends on  $\epsilon$ .

**Example 5.44** The function  $f(x) = x^2$  is uniformly continuous on [0, 1].

**proof:** let  $\epsilon > 0$ , choose  $\delta = \frac{\epsilon}{2}$ , then for all  $x, c \in [0, 1]$  and  $|x - c| < \delta$ , we have  $|x + c| \le 2$ , and hence,

$$|f(x) - f(c)| = |x^2 - c^2| = |x + c||x - c| < |x + c|\delta \le 2\delta = 2 \cdot \epsilon = \epsilon$$

**Remark 5.45** Let  $f: S \to \mathbf{R}$  be a function. We say the function f is not uniformly continuous on S if there exists some  $\epsilon > 0$  such that for all  $\delta > 0$ , there exists some  $x, c \in S$  and  $|x - c| < \delta$  so that  $|f(x) - f(c)| \ge \epsilon$ .

**Example 5.46** The function  $f(x) = \frac{1}{x}$  is not uniformly continuous on (0, 1).

**proof:** choose  $\epsilon = 2$ , let  $\delta > 0$ , choose  $c = \min \{\delta, \frac{1}{2}\}$ ,  $x = \frac{c}{2}$ , then we have

•  $x, c \in (0, 1)$  and  $|x - c| = \frac{c}{2} \le \frac{\delta}{2} < \delta$ 

• 
$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|x||c|} = \frac{c}{2} \cdot \frac{2}{c^2} = \frac{1}{c} \ge 2 = \epsilon$$

**Example 5.47** The function given by  $f(x) = x^2$  is not uniformly continuous on **R**.

**proof:** choose  $\epsilon = 2$ , let  $\delta > 0$ , choose  $c = \frac{2}{\delta}$ ,  $x = c + \frac{\delta}{2}$ , then we have

• 
$$x, c \in \mathbf{R}$$
 and  $|x - c| = \frac{\delta}{2} < \delta$ 

• 
$$|x^2 - c^2| = |x + c||x - c| = (2c + \frac{\delta}{2}) \cdot \frac{\delta}{2} = (\frac{4}{\delta} + \frac{\delta}{2}) \cdot \frac{\delta}{2} = 2 + \frac{\delta^2}{4} \ge 2 = \epsilon$$

**Theorem 5.48** Let  $f: [a, b] \to \mathbf{R}$  be a function. Then, the function f is continuous on [a, b] if and only if f is uniformly continuous on [a, b].

### proof:

- suppose f is uniformly continuous on [a, b]: let  $c \in [a, b]$ ,  $\epsilon > 0$ , then according to uniform continuity,  $\exists \delta > 0$  such that for all  $x \in [a, b]$  and  $|x c| < \delta$ , we have  $|f(x) f(c)| < \epsilon$
- suppose f is continuous on [a, b]
  - assume f is not uniformly continuous on [a, b], then  $\exists \epsilon > 0$  such that  $\forall \delta > 0$ , there exists  $x, c \in [a, b]$  such that  $|x c| < \delta$  but  $|f(x) f(c)| \ge \epsilon$

- choose sequences  $(x_n)_{n=1}^{\infty}$  and  $(c_n)_{n=1}^{\infty}$  such that for all  $n \in \mathbb{N}$ ,  $x_n, c_n \in [a, b]$ ,  $|x_n c_n| < \frac{1}{n}$ , but  $|f(x_n) f(c_n)| \ge \epsilon$
- since  $x_n \in [a, b]$  for all  $n \in \mathbb{N}$ , there exists a subsequence  $(x_{n_i})_{i=1}^{\infty}$  of  $(x_n)_{n=1}^{\infty}$  such that  $x_{n_i} \to c$  and  $c \in [a, b]$  (theorem 3.37)
- take subsequence  $(c_{n_i})_{i=1}^{\infty}$  of  $(c_n)_{n=1}^{\infty}$  according to the indexes  $n_i$  of  $(x_{n_i})_{i=1}^{\infty}$ , then  $c_{n_i} \in [a, b]$  for all  $n \in \mathbb{N} \implies$  there exists a subsequence  $(c_{n_{i_j}})_{j=1}^{\infty}$  such that  $c_{n_{i_j}} \rightarrow d$  and  $d \in [a, b]$
- take subsequence  $(x_{n_{i_j}})_{j=1}^{\infty}$  of  $(x_{n_i})_{i=1}^{\infty}$  according to the indexes  $n_{i_j}$  of  $(c_{n_{i_j}})_{j=1}^{\infty}$ , then  $x_{n_{i_j}} \to c$  since  $x_{n_i} \to c$
- $\begin{array}{l} \ 0 \leq |x_{n_{i_j}} c_{n_{i_j}}| < \frac{1}{n_{i_j}} \text{ and } \frac{1}{n_{i_j}} \to 0 \implies \lim_{j \to \infty} |x_{n_{i_j}} c_{n_{i_j}}| = 0 \implies \\ \lim_{j \to \infty} x_{n_{i_j}} = \lim_{j \to \infty} c_{n_{i_j}} \implies c = d \end{array}$
- since f is continuous on [a,b] and  $x_{n_{i_j}} \to c\text{, } c_{n_{i_j}} \to c\text{, we have}$

$$\lim_{j \to \infty} f(x_{n_{i_j}}) = \lim_{j \to \infty} f(c_{n_{i_j}}) = f(c)$$
$$\implies \quad 0 = |f(c) - f(c)| = \lim_{j \to \infty} |f(x_{n_{i_j}}) - f(c_{n_{i_j}})| \ge \epsilon,$$

which is a contradiction

## Lipschitz continuity

**Definition 5.49** Let  $f: S \to \mathbf{R}$  be a function. We say the function f is **Lipschitz** continuous on S if there exists some  $K \ge 0$  such that for all  $x, y \in S$ , we have  $|f(x) - f(y)| \le K|x - y|$ .

**Remark 5.50** Geometrically, the function f is Lipschitz continuous if and only if all lines intersects the graph of f in at least two distinct points has slope in absolute value less than or equal to K.

**Theorem 5.51** Let  $f: S \to \mathbf{R}$  be a function. If the function f is Lipschitz continuous, then f is uniformly continuous.

**proof:** let  $\epsilon > 0$ 

- f is Lipschitz continuous  $\implies \exists K \geq 0$  such that for all  $x,y \in S,$  we have  $|f(x)-f(y)| \leq K|x-y|$
- choose  $\delta = \epsilon/(K+1)$ , then for all  $x,y \in S$  and  $|x-y| < \delta$ , we have

$$|f(x) - f(y)| \le K|x - y| < K\delta = \frac{K}{K + 1}\epsilon < \epsilon$$

**Example 5.52** The function  $f(x) = \sqrt{x}$  is Lipschitz continuous on  $[1, \infty)$ , but is not Lipschitz continuous on  $[0, \infty)$ .

### proof:

• consider the function  $f: [1, \infty) \to \mathbf{R}$  given by  $f(x) = \sqrt{x}$ , then  $\forall x, y \in [1, \infty)$ : -  $x \ge 1, y \ge 1 \implies \sqrt{x} + \sqrt{y} \ge 2$ 

- hence,

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \le \frac{1}{2}|x - y|$$

 $\implies f$  is Lipschitz continuous with K=1/2

• consider the function  $g\colon [0,\infty)\to {\bf R}$  given by  $g(x)=\sqrt{x}$ , let  $K\ge 0$ , choose  $x=0,\ y=\frac{1}{K^2+1}$ , then

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \left|\frac{\sqrt{x} - \sqrt{y}}{x - y}\right| = \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} = \sqrt{K^2 + 1} > \sqrt{K^2} = K$$
$$\implies |f(x) - f(y)| > K|x - y|$$