4. Series

- series
- Cauchy series
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Series

Definition 4.1 Given a sequence $(x_n)_{n=1}^{\infty}$, the formal object $\sum_{n=1}^{\infty} x_n$ is called a **series**.

A series **converges** if the sequence $(s_m)_{m=1}^{\infty}$ defined by

$$s_m = \sum_{n=1}^{m} x_n = x_1 + \cdots + x_m$$

converges. The numbers s_m are called **partial sums**. If the series converges, we write

$$\sum_{n=1}^{\infty} x_n = \lim_{m \to \infty} s_m.$$

In this case, we treat $\sum_{n=1}^{\infty} x_n$ as a number.

If the sequence $(s_m)_{m=1}^{\infty}$ diverges, we say the series is **divergent**. In this case, $\sum_{n=1}^{\infty} x_n$ is simply a formal object and not a number.

- series need not start at $n = 1$

Example 4.2 The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

proof: the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is given by:

$$\begin{aligned} s_m &= \sum_{n=1}^{m} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{m} \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{m} - \frac{1}{m+1} \\ &= 1 - \frac{1}{m+1}, \end{aligned}$$

hence, $s_m \to 1 \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Theorem 4.3 If $|r| < 1$, then $\sum_{n=0}^{\infty} r^n$ converges and $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

proof:

- the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is given by:

$$s_m = \sum_{n=0}^{m} r^n = \frac{\left(\sum_{n=0}^{m} r^n\right)(1-r)}{1-r} = \frac{\sum_{n=0}^{m}(r^n - r^{n+1})}{1-r} = \frac{1-r^{m+1}}{1-r}$$

- $|r| < 1 \implies r^n \to 0$ (theorem 3.16) $\implies s_m \to \frac{1}{1-r}$

Remark 4.4 Series of the form $\sum_{n=0}^{\infty} \alpha r^n$ with $\alpha, r \in \mathbf{R}$ are called **geometric series**.

Theorem 4.5 Let $(x_n)_{n=1}^{\infty}$ be a sequence and let $M \in \mathbf{N}$. Then, $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=M}^{\infty} x_n$ converges.

proof:

- for all $m \geq M$, we have

$$\sum_{n=1}^{m} x_n = \sum_{n=1}^{M-1} x_n + \sum_{n=M}^{m} x_n$$

- suppose $\sum_{n=1}^{\infty} x_n$ converges, we have

$$\lim_{m\to\infty} \sum_{n=M}^{m} x_n = \lim_{m\to\infty} \left(\sum_{n=1}^{m} x_n - \sum_{n=1}^{M-1} x_n \right) = \lim_{m\to\infty} \left(\sum_{n=1}^{m} x_n \right) - \sum_{n=1}^{M-1} x_n$$

- suppose $\sum_{n=M}^{\infty} x_n$ converges, we have

$$\lim_{m\to\infty} \sum_{n=1}^{m} x_n = \lim_{m\to\infty} \left(\sum_{n=M}^{m} x_n + \sum_{n=1}^{M-1} x_n \right) = \lim_{m\to\infty} \left(\sum_{n=M}^{m} x_n \right) + \sum_{n=1}^{M-1} x_n$$

Cauchy series

Definition 4.6 The series $\sum_{n=1}^{\infty} x_n$ is **Cauchy** if the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is Cauchy.

Theorem 4.7 The series $\sum_{n=1}^{\infty} x_n$ is Cauchy if and only if $\sum_{n=1}^{\infty} x_n$ is convergent.

proof: according to theorem 3.45

- suppose $\sum_{n=1}^{\infty} x_n$ is Cauchy $\implies$ $(s_m)_{m=1}^{\infty}$ is Cauchy $\implies$ $(s_m)_{m=1}^{\infty}$ is convergent $\implies$ $\sum_{n=1}^{\infty} x_n$ is convergent

- suppose $\sum_{n=1}^{\infty} x_n$ is convergent $\implies$ $(s_m)_{m=1}^{\infty}$ is convergent $\implies$ $(s_m)_{m=1}^{\infty}$ is Cauchy $\implies$ $\sum_{n=1}^{\infty} x_n$ is Cauchy

Theorem 4.8 The series $\sum_{n=1}^{\infty} x_n$ is Cauchy if and only if for all $\epsilon > 0$, there exists an $M \in \mathbf{N}$ such that for all $m \geq M$ and $k > m$, we have $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$.

proof: let $\epsilon > 0$

- suppose $\sum_{n=1}^{\infty} x_n$ is Cauchy $\implies$ $(\sum_{n=1}^{m} x_n)_{m=1}^{\infty}$ is Cauchy $\implies$ $\exists M \in \mathbf{N}$ such that $\forall m, k \geq M$ (assume $k > m$), we have

$$\left|\sum_{n=1}^{m} x_n - \sum_{n=1}^{k} x_n\right| < \epsilon \implies \left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$$

- suppose $\exists M \in \mathbf{N}$ such that for all $k > m \geq M$, $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$, then we have

$$\left|\sum_{n=1}^{m} x_n - \sum_{n=1}^{k} x_n\right| = \left|\sum_{n=m+1}^{k} x_n\right| < \epsilon,$$

$i.e.$, $(\sum_{n=1}^{m} x_n)_{m=1}^{\infty}$ is Cauchy $\implies$ $\sum_{n=1}^{\infty} x_n$ is Cauchy

Theorem 4.9 If the series $\sum_{n=1}^{\infty} x_n$ converges then $\lim_{n\to\infty} x_n = 0$.

proof: let $\epsilon > 0$, $\sum_{n=1}^{\infty} x_n$ converges $\implies$ $\sum_{n=1}^{\infty} x_n$ is Cauchy $\implies$ $\exists M_0 \in \mathbf{N}$ such that $\forall k > m \geq M_0$, we have $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$ (theorem 4.8); choose $M = M_0 + 1$, then $\forall m \geq M$, by taking $k = m > m - 1 \geq M_0$, we have

$$|x_m - 0| = |x_m| = \left|\sum_{n=m-1+1}^{m} x_n\right| < \epsilon \implies \lim_{n\to\infty} x_n = 0$$

Remark 4.10 The converse of theorem 4.9 does not hold.

Theorem 4.11 If $|r| \geq 1$ then the series $\sum_{n=0}^{\infty} r^n$ diverges.

proof: If $|r| \geq 1$, then $\lim_{n\to\infty} r^n \neq 0$, according to theorem 4.9, $\sum_{n=0}^{\infty} r^n$ diverges

Corollary 4.12 The series $\sum_{n=0}^{\infty} \alpha r^n$ with $\alpha, r \in \mathbf{R}$ converges if and only if $|r| < 1$.

Theorem 4.13 The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

proof: we show that a subsequence of $(s_m)_{m=1}^{\infty}$ is unbounded

- consider the subsequence $(s_{2^i})_{i=1}^{\infty}$, given by

$$
\begin{aligned}
s_{2^i} &= \sum_{n=1}^{2^i} \frac{1}{n} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{i-1}+1} + \cdots + \frac{1}{2^i}\right) \\
&= 1 + \sum_{k=1}^{i} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n} \\
&\geq 1 + \sum_{k=1}^{i} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{2^k} = 1 + \sum_{k=1}^{i} \frac{1}{2^k}(2^k - (2^{k-1}+1) + 1) \\
&= 1 + \sum_{k=1}^{i} \frac{2^{k-1}}{2^k} = 1 + \sum_{k=1}^{i} \frac{1}{2} = 1 + \frac{i}{2}
\end{aligned}
$$

- $(1 + i/2)_{i=1}^{\infty}$ is unbounded $\implies$ $(s_{2^i})_{i=1}^{\infty}$ is unbounded $\implies$ $(s_m)_{m=1}^{\infty}$ is unbounded $\implies$ $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge

Linearity of series

Theorem 4.14 Let $\alpha \in \mathbf{R}$ and $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series. Then the series $\sum_{n=1}^{\infty}(\alpha x_n + y_n)$ converges and

$$\sum_{n=1}^{\infty}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n.$$

proof: consider the partial sums of $\sum_{n=1}^{\infty}(\alpha x_n + y_n)$, we have

$$\begin{aligned}
& \sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{m} x_n + \sum_{n=1}^{m} y_n \\
\Longrightarrow \quad & \lim_{m\to\infty} \sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \lim_{m\to\infty} \sum_{n=1}^{m} x_n + \lim_{m\to\infty} \sum_{n=1}^{m} y_n \\
\Longrightarrow \quad & \sum_{n=1}^{\infty}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n
\end{aligned}$$

Absolute convergence

Theorem 4.15 If $x_n \geq 0$ for all $n \in \mathbf{N}$, then the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums $(s_m)_{m=1}^{\infty}$ is bounded.

proof:

- suppose $\sum_{n=1}^{\infty} x_n$ converges $\implies$ $(s_m)_{m=1}^{\infty}$ converges $\implies$ $(s_m)_{m=1}^{\infty}$ is bounded
- suppose $(s_m)_{m=1}^{\infty}$ is bounded, since $x_n \geq 0$ for all $n \in \mathbf{N}$, we have

$$s_m = \sum_{n=1}^{m} x_n \leq \sum_{n=1}^{m} x_n + x_{n+1} = s_{m+1},$$

  $i.e.$, $(s_m)_{m=1}^{\infty}$ is monotone increasing $\implies$ $(s_m)_{m=1}^{\infty}$ converges $\implies$ $\sum_{n=1}^{\infty} x_n$ converges

Definition 4.16 The series $\sum_{n=1}^{\infty} x_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |x_n|$ converges.

Theorem 4.17 If the series $\sum_{n=1}^{\infty} x_n$ converges absolutely then $\sum_{n=1}^{\infty} x_n$ converges.

proof:

- we first prove the following claim by induction:

  Lemma 4.18 For all $x_1, \ldots, x_n \in \mathbf{R}$, we have $|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|$.

  - suppose $n = 2$, we have the triangle inequality $|x_1 + x_2| \leq |x_1| + |x_2|$
  - suppose $n > 2$, and $|\sum_{i=1}^{n} x_i| \leq \sum_{i=1}^{n} |x_i|$ holds, we have

$$\left|\sum_{i=1}^{n+1} x_i\right| \leq \left|\sum_{i=1}^{n} x_i\right| + |x_{n+1}| \leq \sum_{i=1}^{n} |x_i| + |x_{n+1}| = \sum_{i=1}^{n+1} |x_i|$$

- $\sum_{n=1}^{\infty} x_n$ converges absolutely $\implies$ $\sum_{n=1}^{\infty} |x_n|$ converges $\implies$ let $\epsilon > 0$, $\exists M \in \mathbf{N}$ s.t. $\forall k > m \geq M$, $|\sum_{n=m+1}^{k} |x_n|| = \sum_{n=m+1}^{k} |x_n| < \epsilon$
- hence, for all $k > m \geq M$, we have $\left|\sum_{n=m+1}^{k} x_n\right| \leq \sum_{n=m+1}^{k} |x_n| < \epsilon \implies$ $\sum_{n=1}^{\infty} x_n$ converges

Remark 4.19 The converse of theorem 4.17 does not hold.

Comparison test

Theorem 4.20 *Comparison test.* Suppose $0 \le x_n \le y_n$ for all $n \in \mathbf{N}$.

- If $\sum_{n=1}^{\infty} y_n$ converges then $\sum_{n=1}^{\infty} x_n$ converges.
- If $\sum_{n=1}^{\infty} x_n$ diverges then $\sum_{n=1}^{\infty} y_n$ diverges.

proof:

- suppose $\sum_{n=1}^{\infty} y_n$ converges $\implies$ $\left(\sum_{n=1}^{m} y_n\right)_{m=1}^{\infty}$ is bounded $\implies$ $\exists B \ge 0$ s.t. $\forall m \in \mathbf{N}$, $\left|\sum_{n=1}^{m} y_n\right| = \sum_{n=1}^{m} y_n \le B$ $\implies$ $\forall m \in \mathbf{N}$, we have
$$0 \le \sum_{n=1}^{m} x_n \le \sum_{n=1}^{m} y_n \le B$$
$\implies$ $\left(\sum_{n=1}^{m} x_n\right)_{m=1}^{\infty}$ is bounded $\implies$ $\sum_{n=1}^{\infty} x_n$ converges (theorem 4.15)
- suppose $\sum_{n=1}^{\infty} x_n$ diverges $\implies$ $\left(\sum_{n=1}^{m} x_n\right)_{m=1}^{\infty}$ is unbounded (theorem 4.15) $\implies$ $\forall B \ge 0$, $\exists m \in \mathbf{N}$ such that $\left|\sum_{n=1}^{m} x_n\right| = \sum_{n=1}^{m} x_n > B$, hence, for this m,
$$\sum_{n=1}^{m} y_n \ge \sum_{n=1}^{m} x_n > B$$
$\implies$ $\left(\sum_{n=1}^{m} y_n\right)_{m=1}^{\infty}$ is unbounded $\implies$ $\sum_{n=1}^{\infty} y_n$ diverges

Theorem 4.21 For $p \in \mathbf{R}$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

proof:

- suppose $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, assume $p \leq 1$, then we have $0 < \frac{1}{n} \leq \frac{1}{n^p}$; the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\implies$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges (theorem 4.20), which is a contradiction
- suppose $p > 1$, let $s_m = \sum_{n=1}^{m} \frac{1}{n^p}$
  - we first show that $s_m \leq s_{2^m}$ for all $m \in \mathbf{N}$: by induction, we have $2^m > m$ for all $m \in \mathbf{N} \implies s_m = \sum_{n=1}^{m} \frac{1}{n^p} \leq \sum_{n=1}^{2^m} \frac{1}{n^p} = s_{2^m}$
  - we now show that s_{2^m} is bounded by $1 + \frac{1}{1-2^{-(p-1)}}$:

$$
\begin{aligned}
s_{2^m} &= \sum_{n=1}^{2^m} \frac{1}{n^p} \\
&= 1 + \left(\frac{1}{2^p}\right) + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \cdots + \left(\frac{1}{(2^{m-1}+1)^p} + \cdots + \frac{1}{(2^m)^p}\right) \\
&= 1 + \sum_{k=1}^{m} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{n^p} \leq 1 + \sum_{k=1}^{m} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{(2^{k-1}+1)^p}
\end{aligned}
$$

$$
\begin{aligned}
&\leq 1 + \sum_{k=1}^{m} \sum_{n=2^{k-1}+1}^{2^k} \frac{1}{(2^{k-1})^p} = 1 + \sum_{k=1}^{m} 2^{-p(k-1)}(2^k - (2^{k-1}+1)+1) \\
&= 1 + \sum_{k=1}^{m} 2^{-(p-1)(k-1)} = 1 + \sum_{k=0}^{m-1} 2^{-(p-1)k} \\
&\leq 1 + \sum_{k=0}^{\infty} 2^{-(p-1)k} = 1 + \sum_{k=0}^{\infty} \left(2^{-(p-1)}\right)^k \\
&= 1 + \frac{1}{1-2^{-(p-1)}},
\end{aligned}
$$

where the last equality is from the fact that $p-1>0$, and using the properties of geometric series (theorem 4.3)

– put together, we have $0 < s_m \leq s_{2^m} \leq 1 + \frac{1}{1-2^{-(p-1)}} \implies (s_m)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies (s_m)_{m=1}^{\infty}$ converges $\implies \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

Ratio test

Theorem 4.22 *Ratio test.* Suppose $x_n \neq 0$ for all n and the limit

$$L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists.

- If $L > 1$ then $\sum_{n=1}^{\infty} x_n$ diverges.
- If $L < 1$ then $\sum_{n=1}^{\infty} x_n$ converges absolutely.

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proof:

- suppose $L > 1$, then $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $\frac{|x_{n+1}|}{|x_n|} \geq 1 \implies \forall n \geq M$, $|x_{n+1}| \geq |x_n| \implies \lim_{n\to\infty} x_n \neq 0 \implies \sum_{n=1}^{\infty} x_n$ diverges (theorem 4.9)
- suppose $L < 1$, let $L < \alpha < 1$
  - $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $\frac{|x_{n+1}|}{|x_n|} \leq \alpha \implies \forall n \geq M$, $|x_{n+1}| \leq \alpha |x_n| \implies$

$$|x_n| \leq \alpha |x_{n-1}| \leq \alpha^2 |x_{n-2}| \leq \cdots \leq \alpha^{n-M} |x_M| \implies |x_n| \leq \alpha^{n-M} |x_M|, \ \forall n \geq M$$

- consider the partial sums of the series $\sum_{n=1}^{\infty} |x_n|$, assume $m > M$, we have

$$\begin{aligned}
\sum_{n=1}^{m} |x_n| &= \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{m} |x_n| \leq \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} |x_n| \\
&\leq \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} \alpha^{n-M} |x_M| = \sum_{n=1}^{M-1} |x_n| + |x_M| \sum_{n=0}^{\infty} \alpha^n \\
&= \sum_{n=1}^{M-1} |x_n| + \frac{|x_M|}{1-\alpha},
\end{aligned}$$

  where the last equality is from the properties of geometric series and $0 < \alpha < 1$

- hence, the sequence of partial sums $(\sum_{n=1}^{m} |x_n|)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies \sum_{n=1}^{\infty} |x_n|$ converges $\implies \sum_{n=1}^{\infty} x_n$ converges absolutely

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Remark 4.23 If $L = 1$ in theorem 4.22 then the test doesn't apply. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Example 4.24 The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ converges absolutely.

proof:

$$\left|\frac{(-1)^n}{n^2+1}\right| = \frac{1}{n^2+1} < \frac{1}{n^2} \implies \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}}{(n+1)^2+1}\right|}{\left|\frac{(-1)^n}{n^2+1}\right|} < \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

Example 4.25 The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbf{R}$.

proof:

$$\lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

Root test

Theorem 4.26 *Root test.* Let $\sum_{n=1}^{\infty} x_n$ be a series and suppose that the limit

$$L = \lim_{n\to\infty} |x_n|^{1/n}$$

exists.

- If $L > 1$ then $\sum_{n=1}^{\infty} x_n$ diverges.
- If $L < 1$ then $\sum_{n=1}^{\infty} x_n$ converges absolutely.

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proof:

- suppose $L > 1$, then $\exists M \in \mathbf{N}$ s.t. $\forall n \geq M$, $|x_n|^{1/n} \geq 1 \implies \forall n \geq M$, $|x_n| \geq 1$ $\implies \lim_{n\to\infty} x_n \neq 0 \implies \sum_{n=1}^{\infty} x_n$ diverges (theorem 4.9)
- suppose $L < 1$, let $L < \alpha < 1$
  - $\exists M \in \mathbf{N}$ such that $\forall n \geq M$, $|x_n|^{1/n} \leq \alpha \implies \forall n \geq M$, $|x_n| \leq \alpha^n$

– consider the partial sums of the series $\sum_{n=1}^{\infty} |x_n|$, assume $m > M$, we have

$$\begin{aligned}
\sum_{n=1}^{m} |x_n| &= \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{m} |x_n| \le \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} |x_n| \\
&\le \sum_{n=1}^{M-1} |x_n| + \sum_{n=M}^{\infty} \alpha^n = \sum_{n=1}^{M-1} |x_n| + \sum_{n=0}^{\infty} \alpha^{M+n} \\
&= \sum_{n=1}^{M-1} |x_n| + \alpha^M \sum_{n=0}^{\infty} \alpha^n \\
&= \sum_{n=1}^{M-1} |x_n| + \frac{\alpha^M}{1-\alpha},
\end{aligned}$$

where the last equality is from the properties of geometric series and $0 < \alpha < 1$

– hence, the sequence of partial sums $(\sum_{n=1}^{m} |x_n|)_{m=1}^{\infty}$ is monotone increasing and bounded $\implies$ $\sum_{n=1}^{\infty} |x_n|$ converges $\implies$ $\sum_{n=1}^{\infty} x_n$ converges absolutely

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Remark 4.27 Similarly, if $L = 1$ in theorem 4.26 then the test doesn't apply.

Alternating series

Theorem 4.28 Let $(x_n)_{n=1}^{\infty}$ be a monotone decreasing sequence with $\lim_{n\to\infty} x_n = 0$. Then the series $\sum_{n=1}^{\infty} (-1)^n x_n$ converges.

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proof: consider the partial sums of $\sum_{n=1}^{\infty} (-1)^n x_n$, given by $s_m = \sum_{n=1}^{m} (-1)^n x_n$

- $(x_n)_{n=1}^{\infty}$ is monotone decreasing and $x_n \to 0 \implies \forall n \in \mathbf{N},\ x_n \geq x_{n+1} \geq 0$
- we first show that the subsequence $(s_{2m})_{m=1}^{\infty}$ converges, notice that

$$s_{2m} = \sum_{n=1}^{2m} (-1)^n x_n = -x_1 + x_2 - x_3 + \cdots - x_{2m-1} + x_{2m} \tag{4.1}$$

  – rearranging the terms in (4.1), since $x_{n+1} \leq x_n$, $\forall n \in \mathbf{N}$, we have

$$\begin{aligned} s_{2m} &= (x_2 - x_1) + (x_4 - x_3) + \cdots + (x_{2m} - x_{2m-1}) \\ &\geq (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{2m} - x_{2m-1}) + (x_{2m+2} - x_{2m+1}) \\ &= s_{2(m+1)} \end{aligned}$$

$\implies (s_{2m})_{m=1}^{\infty}$ is monotone decreasing

- rearranging the terms in (4.1) differently, since $x_n \geq x_{n+1} \geq 0$, $\forall n \in \mathbf{N}$, we have

$$s_{2m} = -x_1 + (x_2 - x_3) + (x_4 - x_5) + \cdots + (x_{2m-2} - x_{2m-1}) + x_{2m} \geq -x_1$$

$\implies$ $(s_{2m})_{m=1}^{\infty}$ is bounded below

- put together, we conclude that $(s_{2m})_{m=1}^{\infty}$ converges, let $s_{2m} \to x$

- we now show that $(s_m)_{m=1}^{\infty}$ also converges to x, let $\epsilon > 0$
  - $s_{2m} \to x \implies \exists M_1 \in \mathbf{N}$ such that $\forall m \geq M_1$, $|s_{2m} - x| < \epsilon/2$
  - $x_n \to 0 \implies \exists M_2 \in \mathbf{N}$ such that $\forall m \geq M_2$, $|x_m| < \epsilon/2$

  let $M = \max\{2M_1 + 1, M_2\}$, then $\forall m \geq M$, $m \geq 2M_1 + 1$ and $m \geq M_2$
  - if m is even $\implies \frac{m}{2} > M_1$, hence

$$|s_m - x| = \left|s_{2.\frac{m}{2}} - x\right| < \epsilon/2 < \epsilon$$

  - if m is odd, then $m-1$ is even and $m - 1 \geq 2M_1 \implies \frac{m-1}{2} \geq M_1$, hence

$$\begin{aligned} |s_m - x| &= |s_{m-1} - x + x_m| = \left|s_{2.\frac{m-1}{2}} - x + x_m\right| \\ &\leq \left|s_{2.\frac{m-1}{2}} - x\right| + |x_m| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

  put together, we have $(s_m)_{m=1}^{\infty}$ converges $\implies \sum_{n=1}^{\infty} (-1)^n x_n$ converges

Corollary 4.29 The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but does not converge absolutely.

proof:

- since $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is monotone decreasing with $\lim_{n\to\infty} \frac{1}{n} = 0$, it follows immediately from theorem 4.28 that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

- since $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not converge absolutely

Rearrangements

Theorem 4.30 Suppose $\sum_{n=1}^{\infty} x_n$ converges absolutely and $\sum_{n=1}^{\infty} x_n = x$. Let $\sigma\colon \mathbf{N} \to \mathbf{N}$ be a bijective function. Then, the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is absolutely convergent and $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$. In other words, absolute convergence implies, if we rearrange the sequence, the new series will still converge to the same value of the original series.

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proof:

- we first show $\sum_{n=1}^{\infty} |x_{\sigma(n)}|$ converges, $i.e.$, $\left(\sum_{n=1}^{m} |x_{\sigma(n)}|\right)_{m=1}^{\infty}$ is bounded
  - $\sum_{n=1}^{\infty} |x_n|$ converges $\implies$ $(\sum_{n=1}^{m} |x_n|)_{m=1}^{\infty}$ is bounded $\implies$ $\exists B \geq 0$ such that $\forall m \in \mathbf{N}$, $\sum_{n=1}^{m} |x_n| \leq B$
  - $\forall m \in \mathbf{N}$, $\{1, \ldots, m\}$ is a finite set $\implies$ $\exists k \in \mathbf{N}$ such that

    $$\sigma(\{1, \ldots, m\}) \subseteq \{1, \ldots, k\},$$

    hence,

    $$\sum_{n=1}^{m} |x_{\sigma(n)}| = \sum_{n \in \sigma(\{1,\ldots,m\})} |x_n| \leq \sum_{n=1}^{k} |x_n| \leq B$$

    $\implies$ $\forall m \in \mathbf{N}$, $\sum_{n=1}^{m} |x_{\sigma(n)}|$ is bounded

- we now show that $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$, let $\epsilon > 0$
  - $\sum_{n=1}^{\infty} x_n = x \implies \exists M_0 \in \mathbf{N}$ such that for all $k > m \geq M_0$, we have

$$\left|\sum_{n=1}^{m} x_n - x\right| < \epsilon/2 \quad \text{and} \quad \left|\sum_{n=m+1}^{k} x_n\right| < \epsilon/2$$

  - the set $\{1, \ldots, M_0\}$ is finite $\implies \exists M \in \mathbf{N}$, $M > M_0$ such that

$$\{1, \ldots, M_0\} \subseteq \sigma(\{1, \ldots, M\}),$$

    hence, for all $m \geq M$, let $p = \max(\sigma(\{1, \ldots, m\})) > M_0$, we have

$$\sigma(\{1, \ldots, m\}) = \{1, \ldots, M_0\} \cup \{M_0 + 1, \ldots, p\}$$

  - consider the partial sums of $\sum_{n=1}^{\infty} x_{\sigma(x)}$, for all $m \geq M$, we have

$$\begin{aligned}\left|\sum_{n=1}^{m} x_{\sigma(n)} - x\right| &= \left|\sum_{n \in \sigma(\{1,\ldots,m\})} x_n - x\right| = \left|\sum_{n=1}^{M_0} x_n - x + \sum_{n=M_0+1}^{p} x_n\right| \\ &\leq \left|\sum_{n=1}^{M_0} x_n - x\right| + \left|\sum_{n=M_0+1}^{p} x_n\right| < \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

$\implies \lim_{m\to\infty} \sum_{n=1}^{m} x_{\sigma(n)} = x \implies \sum_{n=1}^{\infty} x_{\sigma(n)} = x$