Probabilistic Graphical Models

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7. Control as probabilistic inference

- Exact inference
  The graphical model and policy search
  Connection to Bellman equations

- Approximate inference
  Maximum entropy control
  Connection to variational inference
  Obtaining the optimal policy

Outline

- Exact inference
  - The graphical model and policy search
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  - Maximum entropy control
  - Connection to variational inference
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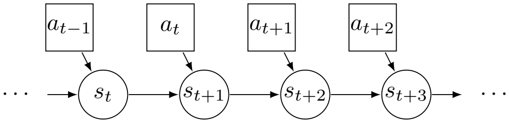
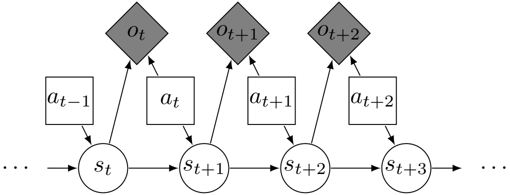
Optimal control problems



$$\text{maximize (over } \theta) \quad \sum_{t=1}^{T} \mathbf{E}_{(s_t,a_t)\sim p(s_t,a_t|\theta)}[r(s_t,a_t)]$$

- T: time horizon
- $p(\tau) = p(s_1, a_1, \ldots, s_T, a_T \mid \theta) = p(s_1) \prod_{t=1}^{T} p(a_t \mid s_t, \theta) p(s_{t+1} \mid s_t, a_t)$

The graphical model



- $\mathcal{O}$: binary random variable, $o_t = 1 \implies$ step t is optimal

$$p(o_t = 1 \mid s_t, a_t) = \exp(r(s_t, a_t))$$

  - assume $r(s_t, a_t) < 0$ for all $s_t \in \mathcal{S}$, $a_t \in \mathcal{A}$

Policy search

target: find the optimal policy $p(a_t \mid s_t, o_{1:T} = \mathbf{1})$

- we will denote $o_{1:T} = \mathbf{1}$ as $o^*_{1:T}$ subsequently for simplicity

- according to the Markov property of the system: $p(a_t \mid s_t, o^*_{1:T}) = p(a_t \mid s_t, o^*_{t:T})$

backward messages

- state-action message

$$\beta(s_t, a_t) = p(o^*_{t:T} \mid s_t, a_t)$$

- state-only message

$$\beta(s_t) = p(o^*_{t:T} \mid s_t)$$

Policy search

- recover $\beta(s_t)$ from $\beta(s_t, a_t)$:

$$\beta(s_t) = p(o^*_{t:T} \mid s_t) = \int_{\mathcal{A}} p(o^*_{t:T} \mid s_t, a_t) p(a_t \mid s_t) \; da_t = \int_{\mathcal{A}} \beta(s_t, a_t) p(a_t \mid s_t) \; da_t$$

  - $p(a_t \mid s_t)$: action prior, assumed to be uniform, i.e., $p(a_t \mid s_t) = \frac{1}{\mathbf{card}(\mathcal{A})}$

- recursive expression

$$\begin{aligned} \beta(s_t, a_t) &= p(o^*_{t:T} \mid s_t, a_t) \\ &= \begin{cases} \exp(r(s_T, a_T)) & t = T \\ \int_{\mathcal{S}} \beta(s_{t+1}) p(s_{t+1} \mid s_t, a_t) p(o^*_t \mid s_t, a_t) \; ds_{t+1} & t < T \end{cases} \end{aligned}$$

Policy search

optimal policy

$$
\begin{aligned}
p(a_t \mid s_t, o^*_{t:T}) &= \frac{p(s_t, a_t \mid o^*_{t:T})}{p(s_t \mid o^*_{t:T})} = \frac{p(o^*_{t:T} \mid s_t, a_t) p(a_t \mid s_t) p(s_t)}{p(o^*_{t:T} \mid s_t) p(s_t)} \\
&\propto \frac{p(o^*_{t:T} \mid s_t, a_t)}{p(o^*_{t:T} \mid s_t)} = \frac{\beta(s_t, a_t)}{\beta(s_t)}
\end{aligned}
$$

- $p(a_t \mid s_t)$ disappears since it's assumed to be uniform

Connection to Bellman equations

backward messages in log-space

$$Q(s_t, a_t) = \log \beta(s_t, a_t)$$

$$V(s_t) = \log \beta(s_t)$$

- marginalization over actions:

$$\beta(s_t) = \int_{\mathcal{A}} \beta(s_t, a_t)\ da_t \implies V(s_t) = \log \int_{\mathcal{A}} \exp(Q(s_t, a_t))\ da_t$$

  - $V(s_t) \approx \max_{a_t} Q(s_t, a_t)$ for large $Q(s_t, a_t)$

Connection to Bellman equations

backups in log-space

$$\beta(s_t, a_t) = \int_{\mathcal{S}} \beta(s_{t+1}) p(s_{t+1} \mid s_t, a_t) p(o_t^* \mid s_t, a_t) \; ds_{t+1}$$

- deterministic dynamics: soft Bellman optimality equations

$$Q(s_t, a_t) = r(s_t, a_t) + V(s_{t+1}) = r(s_t, a_t) + \log \int_{\mathcal{A}} \exp(Q(s_{t+1}, a_{t+1})) \; da_{t+1}$$

- stochastic dynamics:

$$\begin{aligned} Q(s_t, a_t) &= r(s_t, a_t) + \log \int_{\mathcal{S}} p(s_{t+1} \mid s_t, a_t) \exp(V(s_{t+1})) \; ds_{t+1} \\ &= r(s_t, a_t) + \log \mathbf{E}_{s_{t+1} \sim p(s_{t+1} \mid s_t, a_t)} [\exp(V(s_{t+1}))] \end{aligned}$$

  - optimistic Q-functions, creating risk-seeking behavior

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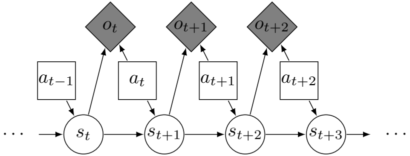
Maximum entropy control

- posterior distribution over trajectories τ given that all actions are optimal:

$$
\begin{aligned}
p(\tau \mid o^*_{1:T}) &\propto p(\tau, o^*_{1:T}) \\
&= p(s_1) \prod_{t=1}^{T} p(o^*_t \mid s_t, a_t) p(s_{t+1} \mid s_t, a_t) \\
&= p(s_1) \prod_{t=1}^{T} \exp(r(s_t, a_t)) p(s_{t+1} \mid s_t, a_t) \\
&= \left[p(s_1) \prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t) \right] \exp\left(\sum_{t=1}^{T} r(s_t, a_t) \right)
\end{aligned}
$$



$$p(o_t = 1 \mid s_t, a_t) = \exp(r(s_t, a_t))$$

- distribution over trajectories τ given some policy π_θ:

$$p_\theta(\tau) = p(s_1) \prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t) \pi_\theta(a_t \mid s_t)$$

Maximum entropy control

the inference problem

$$\text{minimize (over } \theta) \quad D_{\mathrm{KL}}(p_\theta(\tau) \parallel p(\tau \mid o_{1:T}^*))$$

- the optimal policy π^* has to result in a $p^*(\tau)$ that match exactly to the optimal posterior trajectory distribution $p(\tau \mid o_{1:T}^*)$
- $D_{\mathrm{KL}}(p_\theta(\tau) \parallel p(\tau \mid o_{1:T}^*)) = -\mathbf{E}_{\tau \sim p_\theta(\tau)}[\log p(\tau \mid o_{1:T}^*) - \log p_\theta(\tau)]$

Maximum entropy control

$$
\begin{aligned}
-D_{\mathrm{KL}}(p_\theta(\tau) \parallel p(\tau \mid o^*_{1:T})) &= \mathbf{E}_{\tau\sim p_\theta(\tau)}\Bigg[\log p(s_1) + \sum_{t=1}^{T}(\log p(s_{t+1} \mid s_t, a_t) + r(s_t, a_t)) \\
&\qquad - \log p(s_1) - \sum_{t=1}^{T}(\log p(s_{t+1} \mid s_t, a_t) + \log \pi_\theta(a_t \mid s_t))\Bigg] \\
&= \mathbf{E}_{\tau\sim p_\theta(\tau)}\left[\sum_{t=1}^{T} r(s_t, a_t) - \log \pi_\theta(a_t \mid s_t)\right] \\
&= \sum_{t=1}^{T} \mathbf{E}_{(s_t,a_t)\sim p_\theta(s_t,a_t)}[r(s_t, a_t) - \log \pi_\theta(a_t \mid s_t)] \\
&= \sum_{t=1}^{T} \mathbf{E}_{(s_t,a_t)\sim p_\theta(s_t,a_t)}[r(s_t, a_t)] + \sum_{t=1}^{T} \mathbf{E}_{s_t\sim p_\theta(s_t)}[\mathcal{H}(\pi_\theta(s_t))]
\end{aligned}
$$

- $\mathcal{H}(\pi_\theta(s_t))$: the entropy of policy π_θ at state s_t
- minimizing the KL-divergence equals to maximizing the expected reward and the expected policy entropy

Connection to variational inference

variational inference

- approximate some distribution $p(x)$ with another, potentially simpler distribution $q(x)$
- $q(x)$ is taken to be some tractable factorized distribution, which lends itself to tractable exact inference
- approximate inference is performed by optimizing the **variational lower bound** (also called the **evidence lower bound**).

Connection to variational inference

- target distribution

$$p(\tau \mid o^*_{1:T}) = \left[p(s_1)\prod_{t=1}^{T} p(s_{t+1} \mid s_t, a_t)\right] \exp\left(\sum_{t=1}^{T} r(s_t, a_t)\right)$$

- approximate distribution

$$q(\tau) = q(s_1)\prod_{t=1}^{T} q(s_{t+1} \mid s_t, a_t) q(a_t \mid s_t)$$

  - $q(s_1) = p(s_1)$
  - $q(s_{t+1} \mid s_t, a_t) = p(s_{t+1} \mid s_t, a_t)$
  - $q(a_t \mid s_t) = \pi_\theta(a_t \mid s_t)$

Connection to variational inference

- variational lower bound given evidence $o_t = 1$ for all $t = 1, \dots, T$:

$$\begin{aligned}
\log p(o^*_{1:T}) &= \log \iint p(o^*_{1:T}, s_{1:T}, a_{1:T})\ ds_{1:T} da_{1:T} \\
&= \log \iint p(o^*_{1:T}, s_{1:T}, a_{1:T}) \frac{q(s_{1:T}, a_{1:T})}{q(s_{1:T}, a_{1:T})}\ ds_{1:T} da_{1:T} \\
&= \log \mathbf{E}_{(s_{1:T}, a_{1:T}) \sim q(s_{1:T}, a_{1:T})} \left[\frac{p(o^*_{1:T}, s_{1:T}, a_{1:T})}{q(s_{1:T}, a_{1:T})} \right] \\
&\geq \mathbf{E}_{(s_{1:T}, a_{1:T}) \sim q(s_{1:T}, a_{1:T})} [\log p(o^*_{1:T}, s_{1:T}, a_{1:T}) - \log q(s_{1:T}, a_{1:T})] \\
&= \mathbf{E}_{(s_{1:T}, a_{1:T}) \sim q(s_{1:T}, a_{1:T})} \left[\sum_{t=1}^{T} r(s_t, a_t) - \log q(a_t \mid s_t) \right]
\end{aligned}$$

  - the inequality holds because of Jensen's inequality
  - optimizing $\log p(o^*_{1:T})$ equals to optimizing $D_{\mathrm{KL}}(p_\theta(\tau) \parallel p(\tau \mid o^*_{1:T}))$

Obtaining the optimal policy

$$\text{minimize (over } \theta) \quad \sum_{t=1}^{T} \mathbf{E}_{(s_t,a_t)\sim p_\theta(s_t,a_t)}[r(s_t,a_t) - \log \pi_\theta(a_t \mid s_t)]$$

dynamic programming

- the base case:

$$\begin{aligned}
&\mathbf{E}_{(s_T,a_T)\sim p_\theta(s_T,a_T)}[r(s_T,a_T) - \log \pi_\theta(a_T \mid s_T)] \\
&\quad = \mathbf{E}_{(s_T,a_T)\sim p_\theta(s_T,a_T)}\left[\log \frac{\exp(r(s_T,a_T))}{\exp(V(s_T))} - \log \pi_\theta(a_T \mid s_T) + V(s_T)\right] \\
&\quad = \mathbf{E}_{s_T\sim p_\theta(s_T)}\left[-D_{\mathrm{KL}}\left(\pi_\theta(s_T) \,\middle\|\, \frac{1}{\exp(V(s_T))}\exp(r(s_T))\right) + V(s_T)\right]
\end{aligned}$$

  - $V(s_T) = \log \int_{\mathcal{A}} \exp(r(s_T,a_T))\, da_T$: normalizing constant
  - optimal policy: $\pi_\theta(a_T \mid s_T) = \exp(r(s_T,a_T) - V(s_T))$

Obtaining the optimal policy

- the recursive case:

$$
\begin{aligned}
&\mathbf{E}_{(s_t,a_t)\sim p_\theta(s_t,a_t)}[r(s_t,a_t) - \log \pi_\theta(a_t \mid s_t)] + \mathbf{E}_{(s_t,a_t)\sim p_\theta(s_t,a_t)}[\mathbf{E}_{s_{t+1}\sim p(s_{t+1}|s_t,a_t)}[V(s_{t+1})]] \\
&= \mathbf{E}_{(s_t,a_t)\sim p_\theta(s_t,a_t)}[r(s_t,a_t) + \mathbf{E}_{s_{t+1}\sim p(s_{t+1}|s_t,a_t)}[V(s_{t+1})] - \log \pi_\theta(a_t \mid s_t)] \\
&= \mathbf{E}_{(s_t,a_t)\sim p_\theta(s_t,a_t)}\left[\log \frac{\exp(r(s_t,a_t) + \mathbf{E}_{s_{t+1}\sim p(s_{t+1}|s_t,a_t)}[V(s_{t+1})])}{\exp(V(s_t))} - \log \pi_\theta(a_t \mid s_t) + V(s_t)\right] \\
&= \mathbf{E}_{s_t\sim p_\theta(s_t)}\left[-D_{\mathrm{KL}}\left(\pi_\theta(s_t) \,\middle\|\, \frac{1}{\exp(V(s_t))}\exp(Q(s_t))\right) + V(s_t)\right]
\end{aligned}
$$

  - $Q(s_t,a_t) = r(s_t,a_t) + \mathbf{E}_{s_{t+1}\sim p(s_{t+1}|s_t,a_t)}[V(s_{t+1})]$
  - $V(s_t) = \log \int_{\mathcal{A}} \exp(Q(s_t,a_t))\, da_t$
  - optimal policy: $\pi_\theta(a_t \mid s_t) = \exp(Q(s_t,a_t) - V(s_t))$