Reinforcement Learning Journal Club

Algorithms for inverse reinforcement learning

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Inverse reinforcement learning via convex optimization

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February 20, 2025

About this talk

- a (very) brief introduction to convex optimization
- an (old) convex formulation of inverse reinforcement learning (CIRL) problems
  - used for behavioral scientific research: *reward fitting* given subject behavior
  - (sadly) not much applications in engineering
- sloppy math
- examples and opinions (some controversial)

Outline

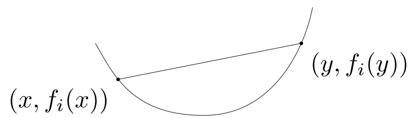
Convex optimization problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- variable $x \in \mathbf{R}^n$
- equality constraints are affine
- $f_0, f_1, \ldots, f_m$ are *convex*: for $0 \leq \theta \leq 1$,

  $$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

  $i.e.$, f_i have *nonnegative (upward) curvature*



why

- effective algorithms, methods (in theory and practice)
- get global solution and *optimality certificate*

Modeling languages for convex optimization

- domain specific languages (DSLs) for convex optimization
  - describe problem in high level human readable language, close to the math
  - can automatically verify problem as convex
  - can automatically transform problem to standard form, then solve

- enables rapid prototyping
- it's now much easier to develop an optimization-based application
- ideal for teaching and research (can do a lot with short scripts)

- gets close to the basic idea: **say what you want, not how to get it**

Implementation

- **CVXPY** (Python): Diamond and Boyd, 2016 [DB16]
- **Convex.jl** (Julia): Udell et al., 2014 [UMZ+14]
- **CVXR** (R): Fu, Narasimhan, and Boyd, 2017 [FNB20]
- **CVX** (Matlab): Grant and Boyd, 2006 [GB14]
- **YALMIP** (Matlab): Lofberg, 2004 [Lof04]

CVXPY example: Non-negative least squares

math:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & x \succeq 0 \end{array}$$

- problem variable: x
- problem data (given): A, b
- $\succeq$: componentwise inequality

CVXPY code:

```
import cvxpy as cp

A, b = ...

x = cp.Variable(n)
obj = cp.norm2(A @ x - b) ** 2
constr = [x >= 0]
prob = cp.Problem(cp.Minimize(obj),
                  constr)
prob.solve()
```

Outline

Introduction to convex optimization

Inverse reinforcement learning via convex optimization

Summary

Preliminaries

Markov decision processes (MDPs)

$$(\mathcal{S},\ \mathcal{A},\ \{P_a\}_{a\in\mathcal{A}},\ r,\ \gamma)$$

- $\mathcal{S}$, $\mathcal{A}$: finite sets of states and actions, with $|\mathcal{S}| = m$, $|\mathcal{A}| = k$
- $\{P_a \in \mathbf{R}_+^{m\times m} \mid P_a\mathbf{1} = \mathbf{1},\ a \in \mathcal{A}\}$: transition probability matrices for all $a \in \mathcal{A}$
- $r \in \mathbf{R}^m$: reward function (or really, vector), which is the *problem variable* and is assumed to be bounded by some positive number $r^{\max} \in \mathbf{R}_{++}$
- $\gamma \in [0, 1)$: discount factor

expert policy $\pi\colon \mathcal{S} \to \mathcal{A}$, assumed to be deterministic with $\pi(s) = a^\star$

value function $v \in \mathbf{R}^m$:

$$v = r + \gamma P_{a^\star} v \implies v = (I - \gamma P_{a^\star})^{-1} r$$

optimality condition

$$
\begin{aligned}
& \pi(s_i) = a^\star \in \operatorname*{argmax}_{a\in\mathcal{A}} p_{i,a}^T v, \quad i = 1,\dots,m \\
\iff \quad & P_{a^\star} v \succeq P_a v, \quad \text{for all } a \in \mathcal{A} \setminus \{a^\star\} \\
\iff \quad & P_{a^\star}(I - \gamma P_{a^\star})^{-1} r \succeq P_a (I - \gamma P_{a^\star})^{-1} r, \quad \text{for all } a \in \mathcal{A} \setminus \{a^\star\} \\
\iff \quad & (P_{a^\star} - P_a)(I - \gamma P_{a^\star})^{-1} r \succeq 0, \quad \text{for all } a \in \mathcal{A} \setminus \{a^\star\},
\end{aligned}
$$

where $p_{i,a}^T$ denotes the ith row of P_a

The CIRL problem

a trivial formulation of CIRL could be the feasibility problem

$$\begin{array}{ll} \text{find} & r \\ \text{subject to} & (P_{a^\star} - P_a)(I - \gamma P_{a^\star})^{-1} r \succeq 0, \quad \text{for all } a \in \mathcal{A} \setminus \{a^\star\} \\ & r^{\max} \succeq r \succeq -r^{\max} \end{array}$$

with variable $r \in \mathbf{R}^m$; data $\{P_a\}_{a \in \mathcal{A}}$, γ; hyperparameter $r^{\max}$

- contains trivial ('meaningless') solutions, *e.g.*, $r = c \in \mathbf{R}^m$ with $c_1 = \cdots = c_m$

to find a 'meaningful' reward, consider

$$\begin{array}{ll} \text{minimize} & J(r) + \lambda \phi(r) \\ \text{subject to} & (P_{a^\star} - P_a)(I - \gamma P_{a^\star})^{-1} r \succeq 0, \quad \text{for all } a \in \mathcal{A} \setminus \{a^\star\} \\ & r^{\max} \succeq r \succeq -r^{\max} \end{array}$$

where $J(r)$, $\phi(r)$ are two criteria that a reward function is considered to be meaningful, and $\lambda \geq 0$ is a hyperparameter

primary objective

$$J(r) = -\sum_{i=1}^{m}\left(p_{i,a^\star}^T v - \sup_{a\in\mathcal{A}\setminus\{a^\star\}} p_{i,a}^T v\right) = -\sum_{i=1}^{m}\inf_{a\in\mathcal{A}\setminus\{a^\star\}}(p_{i,a^\star}^T - p_{i,a}^T)v$$
$$= -\sum_{i=1}^{m}\inf_{a\in\mathcal{A}\setminus\{a^\star\}}\left((p_{i,a^\star}^T - p_{i,a}^T)(I-\gamma P_{a^\star})^{-1}r\right)$$

where $p_{i,a}^T$ denotes the ith row of P_a

- favor reward functions that maximize the margin between the observed expert policy π and all other possible policies at all states

penalty function: ℓ_1-norm, $i.e.$, $\phi(r) = \|r\|_1$

- reward function should be as sparse as possible

put together, we have the convex problem

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^{m}\inf_{a\in\mathcal{A}\setminus\{a^\star\}}\left((p_{i,a^\star}^T - p_{i,a}^T)(I-\gamma P_{a^\star})^{-1}r\right) + \lambda\|r\|_1 \\ \text{subject to} & (P_{a^\star} - P_a)(I-\gamma P_{a^\star})^{-1}r \succeq 0, \quad \text{for all } a\in\mathcal{A}\setminus\{a^\star\} \\ & r^{\max} \succeq r \succeq -r^{\max}, \end{array} \qquad (*)$$

Hyperparameter selection

- trade off between J and ϕ by varying λ in $[0, \infty)$
- there exists a value
$$\lambda^{\max} = \inf_{z \in \partial J(0)} \|z\|_\infty$$
(where $\partial J(0)$ is the subdifferential of $J(r)$ at $r = 0$), such that if $\lambda \geq \lambda^{\max}$, the optimal of (*) is achieved at $r = 0$
  - $\lambda^{\max}$ can be obtained via iterative methods, such as bisection
- find the 'simplest' r given the problem data by setting $\lambda = \lambda^{\max}$

Implementation

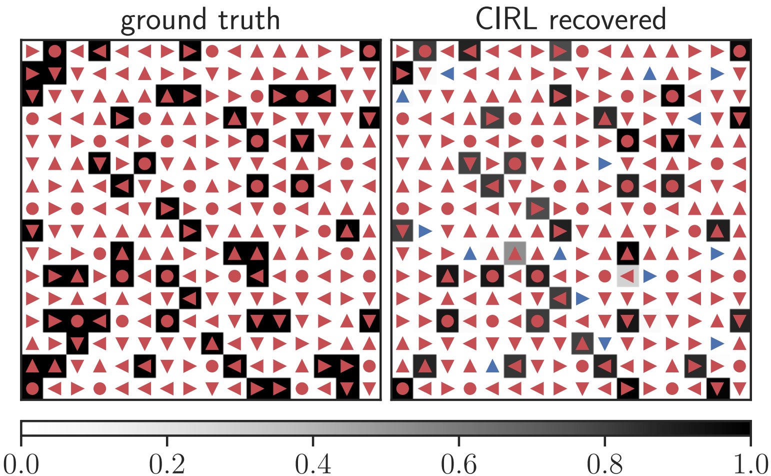
transforming $(*)$ into epigraph form

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s + \lambda \|r\|_1 \\ \text{subject to} & \begin{bmatrix} p_{i,a^\star}^T - p_{i,\tilde{a}_1}^T \\ \vdots \\ p_{i,a^\star}^T - p_{i,\tilde{a}_{k-1}}^T \end{bmatrix} (I - \gamma P_{a^\star})^{-1} r + s_i \succeq 0, \quad i = 1, \ldots, m \\ & (P_{a^\star} - P_{\tilde{a}_i})(I - \gamma P_{a^\star})^{-1} r \succeq 0, \quad i = 1, \ldots, k-1 \\ & r^{\max} \succeq r \succeq -r^{\max} \end{array}$$

- ℓ_1-regularized linear program over variables: $r \in \mathbf{R}^m$ and $s \in \mathbf{R}^m$

```python
import numpy as np
import cvxpy as cp
# problem information (input from user)
m = None  # number of states
gamma = None  # discount factor
Pastr = None  # transition matrix of the optimal action
lPa = []  # list of transition matrices of the other actions
# hyperparameters (input from user)
rmax = None  # reward function bound
lbd = None  # scalarization weight
r = cp.Variable(m)
s = cp.Variable(m)
constraints = []
H = np.linalg.inv(np.identity(m) - gamma * Pastr)
D = np.array([[Pastr[i] - Pa[i] for Pa in lPa] for i in range(m)])
for i in range(m):
    constraints.append(D[i] @ H @ r + s[i] >= 0)
for Pa in lPa:
    constraints.append((Pastr - Pa) @ H @ r >= 0)
constraints.append(rmax >= r)
constraints.append(r >= -rmax)
obj = cp.Minimize(cp.sum(s) + lbd * cp.norm(r, 1))
prob = cp.Problem(obj, constraints)
prob.solve()
```

Example: Gridworld



- $\lambda = 2$, $r^{\max} = 100$
- 0.85 cosine similarity between true and recovered reward
- solved in 1.65 seconds

Outline

Summary

advantages of CIRL formulation:

- global optimality guarantee with certificate
- very easy to implement (though might need some effort to figure out the math)
- very fast for moderate dense problems (can be fast for large problems if sparsity pattern exists in the data)

limitations:

- require deterministic expert policy
- infeasible (or feasible only for $r = 0$) if expert policy is strongly suboptimal

in the paper but not covered here:

- incorporating function approximations ($e.g.$, for continuous $\mathcal{S}$ and $\mathcal{A}$)
- learning from trajectories ($i.e.$, no analytical expert policy available)

Reference

[BV04] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.

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[Lof04] J. Lofberg. YALMIP: A toolbox for modeling and optimization in MATLAB. In *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No. 04CH37508)*, pages 284–289. IEEE, 2004.

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